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Characterizing the interpretation of set theory in Martin-Löf type theory

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Abstract

Constructive Zermelo–Fraenkel set theory, **CZF**, can be interpreted in Martin-Löf type theory via the so-called propositions-as-types interpretation. However, this interpretation validates more than what is provable in **CZF**. We now ask ourselves: is there a reasonably simple axiomatization (by a few axiom schemata say) of the set-theoretic formulae validated in Martin-Löf type theory? The answer is yes for a large collection of statements called the mathematical formulae. The validated mathematical formulae can be axiomatized by suitable forms of the axiom of choice.

The paper builds on a self-interpretation of **CZF** (developed in [M. Rathjen, The formulae-as-classes interpretation of constructive set theory, in: Proof Technology and Computation (Proceedings of the International Summer School Marktoberdorf 2003) IOS Press, Amsterdam, 2004 (in press)]) that provides an "inner" model of **CZF** which also validates the so-called $\Pi\Sigma$-axiom of choice, $\mathbf{\Pi\Sigma}$**–AC**. The crucial technical step taken in the present paper is to investigate the absoluteness properties of this model under the hypothesis $\mathbf{\Pi\Sigma}$**–AC**.

It is also shown that **CZF** plus the $\Pi\Sigma$-axiom of choice possesses the disjunction property, the numerical existence property and the existence property for an important group of formulae.

Keywords: Constructive set theory; Martin-Löf type theory; Formulae-as-classes interpretation; Axiom of choice

1. Introduction

The general topic of Constructive Set Theory (CST) originated in John Myhill's endeavour (see [17]) to discover a simple formalism that relates to Bishop's constructive mathematics as classical Zermelo–Fraenkel Set Theory with the axiom of choice relates to classical Cantorian mathematics. CST provides a standard set theoretical framework for the development of constructive mathematics in the style of Errett Bishop [8]. One of the hallmarks of constructive set theory is that it possesses (cf. [1–3]) a canonical interpretation in Martin-Löf's intuitionistic type theory (see [13,14]) which is considered to be the most acceptable foundational framework of ideas that make precise the constructive approach to mathematics. The interpretation employs the Curry–Howard "propositions-as-types" idea in that the axioms of constructive set theory get interpreted as provably inhabited types.

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The particular system of set theory for which Aczel gave a type-theoretic interpretation is actually a modification of Myhill's system referred to as *Constructive Zermelo–Fraenkel Set Theory*, **CZF**. The interpretation of **CZF** in type theory (notated as $\mathbf{ML_1V}$) not only validates all the theorems of **CZF** but many other interesting set-theoretic statements as well. Ideally, one would like to have a characterization of these statements and determine an extension $\mathbf{CZF}^*$ of **CZF** which deduces exactly the set-theoretic statements validated in the pertaining type theory $\mathbf{ML_1V}$. It will turn out that the search for $\mathbf{CZF}^*$ amounts to finding the "strongest" version of the axiom of choice that is validated in $\mathbf{ML_1V}$. In addition to the axioms of **CZF**, Aczel also interpreted the *Regular Extension Axiom*, **REA**, which ensures the existence of many inductively defined sets. The particular type system that is sufficient for interpreting $\mathbf{CZF}+\mathbf{REA}$ has been denoted by $\mathbf{ML_{1W}V}$. We shall also pursue the question of characterizing the set-theoretic statements validated in $\mathbf{ML_{1W}V}$.

However, rather than giving a characterization of all set-theoretic statements validated in Martin-Löf type theory, we shall restrict attention to a collection of formulae dubbed *mathematical formulae* which includes all the statements of workaday mathematics. The idea behind these formulae is that the sets of ordinary mathematics are of rank $< \omega+\omega$ in the cumulative hierarchy. Roughly speaking, the mathematical formulae are bounded formulae with parameters in $V_{\omega+\omega}$. We shall also consider the wider collection of *generalized mathematical formulae* which from the point of view of **ZFC** is concerned with sets of rank $< \aleph_\omega$. The main results of the paper are expressed in terms of the two choice principles $\mathbf{\Pi\Sigma-AC}$ and $\mathbf{\Pi\Sigma W-AC}$.

Theorem 1.1 (*Cf.* 7.6). *Let ψ be a mathematical sentence and let θ be a generalized mathematical sentence. Then the following hold:*

(i) $\mathbf{CZF}+\mathbf{\Pi\Sigma-AC} \vdash \psi$ *if and only if ψ is validated in* $\mathbf{ML_1V}$.

(ii) $\mathbf{CZF}+\mathbf{REA}+\mathbf{\Pi\Sigma W-AC} \vdash \theta$ *if and only if θ is validated in* $\mathbf{ML_{1W}V}$.

The presentation of constructive mathematics in Martin-Löf type theory is an obvious option for the constructive mathematician. However, it has the drawback that the syntactical apparatus is rather overpowering and that there is no extensive tradition of presenting mathematics in a type theoretic setting. This can be avoided by keeping to the set theoretical language. Constructive set theory is distinctive in that it uses the same language as classical set theory and it thus has the advantage that the ideas, conventions and practice of the set theoretical presentation of ordinary mathematics can be used also in constructive set theory. Theorem 1.1 sheds light on how these two approaches to constructive mathematics are related to each other.

The proof of Theorem 7.6 involves interpretations of $\mathbf{CZF}+\mathbf{\Pi\Sigma-AC}$ in **CZF** and of $\mathbf{CZF}+\mathbf{REA}+\mathbf{\Pi\Sigma W-AC}$ in $\mathbf{CZF}+\mathbf{REA}$, the details of which were presented in [21]. In conjunction with results from [22], we also obtain that $\mathbf{CZF}+\mathbf{\Pi\Sigma-AC}$ and $\mathbf{CZF}+\mathbf{REA}+\mathbf{\Pi\Sigma W-AC}$ have the *disjunction property*, the *numerical existence property* and the *existence property*, not for all formulae, but for the collection of mathematical and generalized mathematical formulae, respectively.

The plan for the paper is as follows: Section 2 discusses choice principles in constructive set theory. After briefly reviewing choice principles which have always featured prominently in constructive accounts of mathematics (axioms of countable choice and dependent choices) we explore the "strongest" versions of choice that can be validated in type theory, notably $\mathbf{\Pi\Sigma-AC}$ and $\mathbf{\Pi\Sigma W-AC}$. Sections 3 and 4 are concerned with interpreting constructive set theory in itself via a formulae-as-classes interpretation. This is done for bounded formulae in Section 3 and for arbitrary formulae in Section 4 via a notion of extended set recursive functions (building on [21]). Section 5 deals with the question of how the formulae-as-classes interpretation can be characterized via an inner model construction on the basis of $\mathbf{\Pi\Sigma-AC}$ and $\mathbf{\Pi\Sigma W-AC}$, respectively. Section 6 features interpretations of type theory in set theory also drawing on the notion of extended set recursive functions. In Section 7 we are in a position to prove the main result Theorem 1.1. The last section presents some results about existential definability in theories with $\mathbf{\Pi\Sigma-AC}$ and $\mathbf{\Pi\Sigma W-AC}$.

Notation. We will use $\langle x, y\rangle$ to notate the ordered pair of x and y. We use $\mathrm{Fun}(f)$ to express that f is a function. $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ denote the domain and the range of f, respectively. $f : A \to B$ is used to convey that f is a function with $\mathrm{dom}(f) = A$ and $\mathrm{ran}(f) \subseteq B$.

2. The axiom of choice in constructive set theories

Among the axioms of set theory, the axiom of choice is distinguished by the fact that is it the only one that one finds mentioned in workaday mathematics. In the mathematical world of the beginning of the 20th century, discussions about the status of the axiom of choice were important. In 1904 Zermelo proved that every set can be well-ordered by employing the axiom of choice. While Zermelo argued that it was self-evident, it was also criticized as an excessively non-constructive principle by some of the most distinguished analysts of the day, notably Borel, Baire, and Lebesgue. At first blush this reaction against the axiom of choice utilized in Cantor's new theory of sets is surprising as the French analysts had used and continued to use choice principles routinely in their work. However, in the context of 19th century classical analysis only the Axiom of Dependent Choices, **DC**, is invoked and considered to be natural, while the full axiom of choice is unnecessary and even has some counterintuitive consequences.

Unsurprisingly, the axiom of choice does not have an unambiguous status in constructive mathematics either. On the one hand it is said to be an immediate consequence of the constructive interpretation of the quantifiers. Any proof of $\forall x \in A\, \exists y \in B\, \phi(x, y)$ must yield a function $f : A \to B$ such that $\forall x \in A\, \phi(x, f(x))$. This is certainly the case in Martin-Löf's intuitionistic theory of types. On the other hand, it has been observed that the full axiom of choice cannot be added to systems of extensional constructive set theory without yielding constructively unacceptable cases of excluded middle (see [9]). In extensional intuitionistic set theories, a proof of a statement $\forall x \in A\, \exists y \in B\, \phi(x, y)$, in general, provides only a function F, which when fed a proof p witnessing $x \in A$, yields $F(p) \in B$ and $\phi(x, F(p))$. Therefore, in the main, such an F cannot be rendered a function of x alone. Choice will then hold over sets which have a canonical proof function, where a constructive function h is a canonical proof function for A if for each $x \in A$, $h(x)$ is a constructive proof that $x \in A$. Such sets having natural canonical proof functions "built in" have been called *bases* (cf. [24], p. 841).

The particular form of constructivism adhered to in this paper is Martin-Löf's intuitionistic type theory (cf. [13, 14]). Set-theoretic choice principles will be considered as constructively justified if they can be shown to hold in the interpretation in type theory. Moreover, looking at set theory from a type-theoretic point of view has turned out to be a valuable heuristic tool for finding new constructive choice principles. For more information on choice principles in the constructive context see [20].

2.1. Some constructive choice principles

In many a text on constructive mathematics, axioms of countable choice and dependent choices are accepted as constructive principles. This is, for instance, the case in Bishop's constructive mathematics (cf. [8]) as well as Brouwer's intuitionistic analysis (cf. [24], Ch. 4, Sect. 2). Myhill also incorporated these axioms in his constructive set theory [17].

The weakest constructive choice principle we shall consider is the *Axiom of Countable Choice*, $\mathbf{AC}_\omega$, i.e. whenever F is a function with domain ω such that $\forall i \in \omega\, \exists y \in F(i)$, then there exists a function f with domain ω such that $\forall i \in \omega\; f(i) \in F(i)$.

A mathematically very useful axiom to have in set theory is the *Dependent Choices Axiom*, **DC**, i.e., for all formulae ψ, whenever

$$(\forall x \in a)\, (\exists y \in a)\, \psi(x, y)$$

and $b_0 \in a$, then there exists a function $f : \omega \to a$ such that $f(0) = b_0$ and

$$(\forall n \in \omega)\, \psi(f(n), f(n+1)).$$

Even more useful is the *Relativized Dependent Choices Axiom*, **RDC**. It asserts that for arbitrary formulae ϕ and ψ, whenever

$$\forall x\big[\phi(x) \to \exists y\big(\phi(y) \wedge \psi(x, y)\big)\big]$$

and $\phi(b_0)$, then there exists a function f with domain ω such that $f(0) = b_0$ and

$$(\forall n \in \omega)\big[\phi(f(n)) \wedge \psi(f(n), f(n+1))\big].$$

2.2. Operations on sets

The interpretation of constructive set theory in type theory not only validates all the theorems of **CZF** (resp. **CZF**+ **REA**) but many other interesting set-theoretic statements, including several new choice principles which will be described next. To state these principles we need to introduce various operations on classes.

Remark 2.1 (*Class Notation*). In doing mathematics in **CZF** we shall exploit the use of class notation and terminology, just as in classical set theory. Given a formula $\phi(x)$ there may not exist a set of the form $\{x : \phi(x)\}$. But there is nothing wrong with thinking about such collection. So, if $\phi(x)$ is a formula in the language of set theory we may form a class $\{x : \phi(x)\}$. We allow $\phi(x)$ to have free variables other than x, which are considered parameters upon which the class depends. Informally, we call any collection of the form $\{x : \phi(x)\}$ a *class*. However formally, classes do not exist, and expressions involving them must be thought of as abbreviations for expressions not involving them. Classes A, B are defined to be equal if $\forall x[x \in A \leftrightarrow x \in B]$.

We may also consider an augmentation of the language of set theory whereby we allow atomic formulae of the form $y \in A$ and $A = B$ with A, B being classes. There is no harm in taking such liberties as any such formula can be translated back into the official language of set theory by re-writing $y \in \{x : \phi(x)\}$ and $\{x : \phi(x)\} = \{y : \psi(y)\}$ as $\phi(y)$ and $\forall z\,[\phi(z) \leftrightarrow \psi(z)]$, respectively (with z not in $\phi(x)$ and $\psi(y)$).

Definition 2.2. Let $\mathbf{CZF}_{Exp}$ denote the modification of **CZF** with Exponentiation in place of Subset Collection.

Remark 2.3. In all the results of this paper, **CZF** could be replaced by $\mathbf{CZF}_{Exp}$, that is to say, for the purposes of this paper it is enough to assume Exponentiation rather than Subset Collection. However, in what follows we shall not point this out again.

Definition 2.4 (**CZF**). If A is a set and B_x are classes for all $x \in A$, we define a class $\prod_{x\in A} B_x$ by:

$$\prod_{x\in A} B_x := \left\{ f : A \to \bigcup_{x\in A} B_x \mid \forall x \in A(f(x) \in B_x) \right\}. \tag{1}$$

If A is a class and B_x are classes for all $x \in A$, we define a class $\sum_{x\in A} B_x$ by:

$$\sum_{x\in A} B_x := \{\langle x, y\rangle \mid x \in A \wedge y \in B_x\}. \tag{2}$$

If A is a class and a, b are sets, we define a class $\mathbf{I}(A, a, b)$ by:

$$\mathbf{I}(A, a, b) := \{z \in 1 \mid a = b \wedge a, b \in A\}. \tag{3}$$

If A is a class and for each $a \in A$, B_a is a set, then

$$\mathbf{W}_{a\in A} B_a$$

is the smallest class Y such that whenever $a \in A$ and $f : B_a \to Y$, then $\langle a, f\rangle \in Y$.

Lemma 2.5 (**CZF**). *If A,B,a,b are sets and B_x are sets for all $x \in A$, then $\prod_{x\in A} B_x$, $\sum_{x\in A} B_x$ and $\mathbf{I}(A, a, b)$ are sets.*

Proof. First of all, we need to prove that $\bigcup_{x\in A} B_x$ is a set. Indeed, $g = \{\{x, \{x, B_x\}\} \mid x \in A\}$, and so $\bigcup\bigcup g = \{z, x, B_x \mid z \in x, x \in A\}$ is a set by Union. Now

$$\mathrm{ran}(g) = \left\{ y \in \bigcup\bigcup g \mid \exists x \in \bigcup\bigcup g\, (\langle x, y\rangle \in g) \right\}$$

and $\bigcup_{x\in A} B_x = \bigcup \mathrm{ran}(g)$ are sets by Bounded Separation and Union.
1: The class of all functions from A to $\bigcup_{x\in A} B_x$ is a set by Exponentiation and

$$\prod_{x\in A} B_x := \left\{ f : A \to \bigcup_{x\in A} B_x \mid \forall x \in A(f(x) \in B_x) \right\}$$

is a set by Bounded Separation, since $\forall x \in A(f(x) \in B_x)$ can be rewritten as

$$\forall x \in A\, \exists y \in \mathrm{ran}(f) \exists y' \in \mathrm{ran}(g)(\langle x, y\rangle \in f \wedge \langle x, y'\rangle \in g \wedge y \in y').$$

2: Using from above that $\bigcup_{x\in A} B_x$ is a set, by Pairing, Union and Replacement we obtain a set

$$A \times \bigcup_{x\in A} B_x = \left\{\langle x, y\rangle \mid x \in A \wedge y \in \bigcup_{x\in A} B_x\right\}.$$

Now, the set

$$\sum_{x\in A} B_x := \left\{\langle x, y\rangle \in A \times \bigcup_{x\in A} B_x \mid x \in A \wedge y \in B_x\right\}$$

exists by Bounded Separation, since $x \in A \wedge y \in B_x$ can be rewritten as

$$x \in A \wedge \exists y' \in \mathrm{ran}(g)(\langle x, y'\rangle \in g \wedge y \in y').$$

3: $\mathbf{I}(A, a, b)$ is a set by Bounded Separation. □

Lemma 2.6 (**CZF** + **REA**). *If A is a set and B_x is a set for all $x \in A$, then $\mathbf{W}_{a\in A} B_a$ is a set.*

Proof. This follows from [3], Corollary 5.3. □

2.3. Inductively defined classes

In the following we shall introduce several inductively defined classes, and, moreover, we have to ensure that such classes can be formalized in **CZF**.

We define an *inductive definition* to be a class of ordered pairs. If Φ is an inductive definition and $\langle x, a\rangle \in \Phi$ then we write

$$\frac{x}{a}\ \Phi$$

and call $\frac{x}{a}$ an *(inference) step* of Φ, with set x of *premisses* and *conclusion* a. For any class Y, let

$$\Gamma_\Phi(Y) = \left\{a \mid \exists x \left(x \subseteq Y \quad\wedge\quad \frac{x}{a}\ \Phi\right)\right\}.$$

The class Y is *Φ-closed* if $\Gamma_\Phi(Y) \subseteq Y$. Note that Γ is monotone; i.e. for classes Y_1, Y_2, whenever $Y_1 \subseteq Y_2$, then $\Gamma(Y_1) \subseteq \Gamma(Y_2)$.

We define the class *inductively defined by Φ* to be the smallest Φ-closed class. The main result about inductively defined classes states that this class, denoted $\mathbf{I}(\Phi)$, always exists.

Lemma 2.7 (**CZF**) (*Class Inductive Definition Theorem*). *For any inductive definition Φ there is a smallest Φ-closed class $\mathbf{I}(\Phi)$.*

Moreover, call a set G of ordered pairs good *if*

$$\langle a, y\rangle \in G \Rightarrow y \in \Gamma_\Phi(G^{\in a}) \qquad (*)$$

where

$$G^{\in a} = \{y' \mid \exists x{\in}a\ \langle x, y'\rangle \in G\}.$$

Letting $J = \bigcup\{G \mid G \text{ is good}\}$ and $J^a = \{x \mid \langle a, x\rangle \in J\}$, it holds

$$\mathbf{I}(\Phi) = \bigcup_a J^a,$$

and for each a,

$$J^a = \Gamma_\Phi\left(\bigcup_{x\in a} J^x\right).$$

J is uniquely determined by the above, and its stages J^a will be denoted by Γ_Φ^a.

Proof. [2], section 4.2 or [4], Theorem 5.1. □

Lemma 2.8 (**CZF**). *There exists a smallest* $\mathbf{\Pi\Sigma}$*-closed class, i.e., a smallest class* $\mathbf{Y}$ *such that the following hold:*

(i) $n \in \mathbf{Y}$ *for all* $n \in \omega$*;*
(ii) $\omega \in \mathbf{Y}$*;*
(iii) $\prod_{x\in A} B_x \in \mathbf{Y}$ *and* $\sum_{x\in A} B_x \in \mathbf{Y}$ *whenever* $A \in \mathbf{Y}$ *and* $B_x \in \mathbf{Y}$ *for all* $x \in A$.

Likewise, there exists a smallest $\mathbf{\Pi\Sigma I}$*-closed class, i.e. a smallest class* $\mathbf{Y}^*$*, which, in addition to the closure conditions* (i)–(iii) *above, satisfies:*

(iv) $\mathbf{I}(A, a, b) \in \mathbf{Y}^*$ *whenever* $A \in \mathbf{Y}^*$ *and* $a, b \in A$.

Proof. The classes $\mathbf{Y}$ and $\mathbf{Y}^*$ are inductively defined, and therefore exist by Lemma 2.7. To be precise, the respective inductive definitions of these classes are given by the classes $\Phi_1, \ldots, \Phi_5$ consisting of the following pairs:

(i) $\dfrac{\ }{n}\ \Phi_1$, for all $n \in \omega$;
(ii) $\dfrac{\ }{\omega}\ \Phi_2$;
(iii) $\dfrac{\{\mathrm{dom}(g)\} \cup \mathrm{ran}(g)}{\prod_{x\in A} g(x)}\ \Phi_3$, for all functions g with $\mathrm{dom}(g) = A$;
(iv) $\dfrac{\{\mathrm{dom}(g)\} \cup \mathrm{ran}(g)}{\sum_{x\in A} g(x)}\ \Phi_4$, for all functions g with $\mathrm{dom}(g) = A$;
(v) $\dfrac{\{A\}}{\mathbf{I}(A, a, b)}\ \Phi_5$, if $a, b \in A$.

(Clause (v) is only needed to define $\mathbf{Y}^*$.) □

Lemma 2.9 (**CZF + REA**). *There exists a least* $\mathbf{\Pi\Sigma W}$*-closed class, i.e. a smallest class* $\mathbf{Y_w}$ *that in addition to the clauses* (i), (ii), (iii) *of Lemma* 2.8 *satisfies:*

(vi) $\mathbf{W}_{a\in A} B_a \in \mathbf{Y_w}$ *whenever* $A \in \mathbf{Y_w}$ *and* $B_x \in \mathbf{Y_w}$ *for all* $x \in A$.

Likewise, there exists a smallest $\mathbf{\Pi\Sigma WI}$*-closed class, i.e. a least class* $\mathbf{Y}^*_\mathbf{w}$*, which, in addition to the closure conditions above, satisfies clause* (iv) *of Lemma* 2.8*.*

Proof. Virtually the same as for Lemma 2.8. □

2.4. Strong choice principles

Definition 2.10. The $\mathbf{\Pi\Sigma}$-generated sets are the sets in the smallest $\mathbf{\Pi\Sigma}$-closed class, i.e. $\mathbf{Y}$. Similarly one defines the $\mathbf{\Pi\Sigma I}$, $\mathbf{\Pi\Sigma W}$ and $\mathbf{\Pi\Sigma WI}$-generated sets.

A set P is a *base* if for any P-indexed family $(X_a)_{a\in P}$ of inhabited sets X_a, there exists a function f with domain P such that, for all $a \in P$, $f(a) \in X_a$.

$\mathbf{\Pi\Sigma-AC}$ is the statement that every $\Pi\Sigma$-generated set is a base. Similarly one defines the axioms $\mathbf{\Pi\Sigma I-AC}$, $\mathbf{\Pi\Sigma WI-AC}$, and $\mathbf{\Pi\Sigma W-AC}$.

Lemma 2.11.

(i) (**CZF**) *For every* $A \in \mathbf{Y}^*$ *there exists a* $B \in \mathbf{Y}$ *with a bijection* $h : B \to A$.
(ii) (**CZF + REA**) *For every* $A \in \mathbf{Y}^*_\mathbf{w}$ *there exists a* $B \in \mathbf{Y_w}$ *with a bijection* $h : B \to A$.

Proof. See the lemma following Theorem 3.7 in [3]. □

Corollary 2.12.

(i) (**CZF**) $\mathbf{\Pi\Sigma-AC}$ *and* $\mathbf{\Pi\Sigma I-AC}$ *are equivalent.*
(ii) (**CZF + REA**) $\mathbf{\Pi\Sigma W-AC}$ *and* $\mathbf{\Pi\Sigma WI-AC}$ *are equivalent.*

Proof. $\mathbf{\Pi\Sigma I{-}AC}$ obviously implies $\mathbf{\Pi\Sigma{-}AC}$, since $\mathbf{Y} \subseteq \mathbf{Y}^*$. To prove the converse, assume $\mathbf{\Pi\Sigma{-}AC}$, $A \in \mathbf{Y}^*$, and $\forall x \in A \exists y \varphi(x, y)$, where φ is a formula of **CZF**. Take a B and a bijection $h : A \to B$ which exists by the previous Lemma; then $\forall x \in B \exists y\, \varphi(h^{-1}(x), y)$. By $\mathbf{\Pi\Sigma{-}AC}$,

$$\exists f : B \to \mathrm{V}\, \forall x \in B\, \varphi(h^{-1}(x),\ f(x)).$$

This yields

$$\forall x \in A\, \varphi(h^{-1} \circ h(x),\ f \circ h(x))$$

so that $\forall x \in A\, \varphi(x,\ f \circ h(x))$.

The proof of (ii) is similar. □

3. Interpreting bounded formulae as sets

Notation. For sets x and y, we define $\sup(x, y)$ as $\langle x, y\rangle$. If $\alpha = \sup(A, f)$, where f is a function with domain A, we define $\bar{\alpha} := A$ and $\tilde{\alpha} := f$.

Definition 3.1 (**CZF**). By Lemma 2.7 we define classes $\mathbf{V}(\mathbf{Y}^*)$ and $\mathbf{H}(\mathbf{Y}^*)$ by the following rules:

$$\frac{a \in \mathbf{Y}^* \qquad f : a \to \mathbf{V}(\mathbf{Y}^*)}{\sup(a,\ f) \in \mathbf{V}(\mathbf{Y}^*)}, \tag{4}$$

$$\frac{a \in \mathbf{Y}^* \qquad f : a \to \mathbf{H}(\mathbf{Y}^*)}{\mathrm{ran}(f) \in \mathbf{H}(\mathbf{Y}^*)}. \tag{5}$$

The classes $\mathbf{V}(\mathbf{Y})$ and $\mathbf{H}(\mathbf{Y})$ are defined in the same vein by replacing $\mathbf{Y}^*$ by $\mathbf{Y}$ in the foregoing clauses.

Definition 3.2 (**CZF**). The (class) functions $\dot{=} : \mathbf{V}(\mathbf{Y}^*) \times \mathbf{V}(\mathbf{Y}^*) \to \mathbf{Y}^*$ and $\dot{\in} : \mathbf{V}(\mathbf{Y}^*) \times \mathbf{V}(\mathbf{Y}^*) \to \mathbf{Y}^*$ are defined by recursion as follows:

$$\dot{=}(\alpha, \beta) \quad \text{is} \quad \prod_{x \in \bar{\alpha}} \sum_{y \in \bar{\beta}} \dot{=}(\tilde{\alpha}(x), \tilde{\beta}(y)) \times \prod_{y \in \bar{\beta}} \sum_{x \in \bar{\alpha}} \dot{=}(\tilde{\alpha}(x), \tilde{\beta}(y)), \tag{6}$$

$$\dot{\in}(\alpha, \beta) \quad \text{is} \quad \sum_{y \in \bar{\beta}} \dot{=}(\alpha, \tilde{\beta}(y)). \tag{7}$$

Definition 3.3 ($\mathbf{CZF} + \mathbf{REA}$). In the same vein as in Definitions 3.1 and 3.2 we define classes $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}})$, $\mathbf{V}(\mathbf{Y}_{\mathbf{w}})$, $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$, $\mathbf{H}(\mathbf{Y}_{\mathbf{w}})$, and (class) functions $\dot{=} : \mathbf{V}(\mathbf{Y}^*_{\mathbf{w}}) \times \mathbf{V}(\mathbf{Y}^*_{\mathbf{w}}) \to \mathbf{Y}^*_{\mathbf{w}}$ and $\dot{\in} : \mathbf{V}(\mathbf{Y}^*_{\mathbf{w}}) \times \mathbf{V}(\mathbf{Y}^*_{\mathbf{w}}) \to \mathbf{Y}^*_{\mathbf{w}}$ by replacing $\mathbf{Y}^*$ with $\mathbf{Y}^*_{\mathbf{w}}$.

Convention. We will write $\alpha \dot{=} \beta$ and $\alpha \dot{\in} \beta$ for $\dot{=}(\alpha, \beta)$ and $\dot{\in}(\alpha, \beta)$, respectively.

Lemma 3.4.

(i) (**CZF**) $\mathbf{H}(\mathbf{Y}) = \mathbf{H}(\mathbf{Y}^*)$.
(ii) ($\mathbf{CZF} + \mathbf{REA}$) $\mathbf{H}(\mathbf{Y}_{\mathbf{w}}) = \mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$.

Proof. (i): Plainly, we have $\mathbf{H}(\mathbf{Y}) \subseteq \mathbf{H}(\mathbf{Y}^*)$. To show $\mathbf{H}(\mathbf{Y}^*) \subseteq \mathbf{H}(\mathbf{Y})$, we shall draw on Lemma 2.7. Let $\Gamma_{\mathbf{H}(\mathbf{Y}^*)}$ be the operator that inductively defines $\mathbf{H}(\mathbf{Y}^*)$ so that

$$\mathbf{H}(\mathbf{Y}^*) = \bigcup_a \Gamma^a_{\mathbf{H}(\mathbf{Y}^*)}.$$

Proceeding by set induction on a, we show that $\Gamma^a_{\mathbf{H}(\mathbf{Y}^*)} \subseteq \mathbf{H}(\mathbf{Y})$. So assume that $\bigcup_{b \in a} \Gamma^b_{\mathbf{H}(\mathbf{Y}^*)} \subseteq \mathbf{H}(\mathbf{Y})$ and suppose $g : A \to \bigcup_{b \in a} \Gamma^b_{\mathbf{H}(\mathbf{Y}^*)}$, where $A \in \mathbf{Y}^*$. Owing to Lemma 2.11 there exists $B \in \mathbf{Y}$ and a bijection $h : B \to A$. Then we have $g \circ h : \mathbf{Y} \to \mathbf{H}(\mathbf{Y})$, and thus $\mathrm{ran}(g) = \mathrm{ran}(g \circ h) \in \mathbf{H}(\mathbf{Y})$. Consequently, $\Gamma^a_{\mathbf{H}(\mathbf{Y}^*)} \subseteq \mathbf{H}(\mathbf{Y})$.
(ii) is proved similarly. □

Definition 3.5.

(i) (**CZF**) The mapping $\ell : \mathbf{V}(\mathbf{Y}^*) \to \mathbf{H}(\mathbf{Y}^*)$ is defined by recursion via

$$\ell(\sup(a, f)) := \{\ell(f(i)) \mid i \in a\} = \mathrm{ran}(\ell \circ f). \tag{8}$$

(ii) (**CZF** + **REA**) The mapping $\ell_{\mathbf{w}} : \mathbf{V}(\mathbf{Y}^*_{\mathbf{w}}) \to \mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$ is defined by recursion via

$$\ell_{\mathbf{w}}(\sup(a, f)) := \{\ell_{\mathbf{w}}(f(i)) \mid i \in a\} = \mathrm{ran}(\ell_{\mathbf{w}} \circ f). \tag{9}$$

Lemma 3.6.

(i) ($\mathbf{CZF} + \mathbf{\Pi\Sigma{-}AC}$) *$\ell$ is surjective.*
(ii) ($\mathbf{CZF} + \mathbf{\Pi\Sigma W{-}AC}$) *$\ell_{\mathbf{w}}$ is surjective.*

Proof. By induction on the inductive generation of $\mathbf{H}(\mathbf{Y}^*)$, we prove

$$\forall x \in \mathbf{H}(\mathbf{Y}^*)\exists z \in \mathbf{V}(\mathbf{Y}^*)(\ell(z) = x).$$

Let $x \in \mathbf{H}(\mathbf{Y}^*)$. Then $x = \mathrm{ran}(g)$ for some function $g : A \to \mathbf{H}(\mathbf{Y}^*)$, where $A \in \mathbf{Y}^*$. Inductively we have $\forall u \in A\exists c \in \mathbf{V}(\mathbf{Y}^*)(\ell(c) = g(u))$. By $\mathbf{\Pi\Sigma I - AC}$, which by Corollary 2.12 is equivalent to $\mathbf{\Pi\Sigma - AC}$, there is a function $f : A \to \mathbf{V}(\mathbf{Y}^*)$ such that $\forall u \in A(\ell(f(u)) = g(u))$. Note that $\sup(A, f) \in \mathbf{V}(\mathbf{Y}^*)$. Hence $\ell(\sup(A, f)) = \mathrm{ran}(\ell \circ f) = \mathrm{ran}(g) = x$.
(ii) is proved similarly. □

Lemma 3.7. ($\mathbf{CZF} + \mathbf{\Pi\Sigma{-}AC}$) *Let $\alpha, \beta \in \mathbf{V}(\mathbf{Y}^*)$. Then we have:*

(i) $\exists i \in (\alpha \doteq \beta) \;\Leftrightarrow\; \ell(\alpha) = \ell(\beta)$.
(ii) $\exists i \in (\alpha \mathrel{\dot\in} \beta) \;\Leftrightarrow\; \ell(\alpha) \in \ell(\beta)$.

Proof. (i) is proved by induction on α and β:

$$\begin{aligned}
& \exists i \in (\alpha \doteq \beta) \\
\Longleftrightarrow\quad & \exists uv(\langle u, v\rangle \in (\alpha \doteq \beta)) \\
\Longleftrightarrow\quad & \exists u \in \prod_{x\in\bar\alpha}\sum_{y\in\bar\beta}(\tilde\alpha(x) \doteq \tilde\beta(y)) \wedge \exists v \in \prod_{y\in\bar\beta}\sum_{x\in\bar\alpha}(\tilde\alpha(x) \doteq \tilde\beta(y)) \\
\overset{\mathbf{\Pi\Sigma I-AC}}{\Longleftrightarrow}\quad & \forall x \in \bar\alpha\exists y \in \bar\beta\exists i \in (\tilde\alpha(x) \doteq \tilde\beta(y)) \wedge \forall y \in \bar\beta\exists x \in \bar\alpha\exists j \in (\tilde\alpha(x) \doteq \tilde\beta(y)) \\
\overset{\mathrm{IH}}{\Longleftrightarrow}\quad & \forall x \in \bar\alpha\exists y \in \bar\beta(\ell(\tilde\alpha(x)) = \ell(\tilde\beta(y))) \wedge \forall y \in \bar\beta\exists x \in \bar\alpha(\ell(\tilde\alpha(x)) = \ell(\tilde\beta(y))) \\
\Longleftrightarrow\quad & \mathrm{ran}(\ell \circ \tilde\alpha) = \mathrm{ran}(\ell \circ \tilde\beta) \\
\Longleftrightarrow\quad & \ell(\alpha) = \ell(\beta).
\end{aligned}$$

(ii) now follows from (i):

$$\begin{aligned}
\exists i \in (\alpha \mathrel{\dot\in} \beta) \;&\Longleftrightarrow\; \exists y \in \bar\beta\exists j \in (\alpha \doteq \tilde\beta(y)) \\
&\overset{\mathrm{(i)}}{\Longleftrightarrow}\; \exists y \in \bar\beta(\ell(\alpha) = \ell(\tilde\beta(y))) \\
&\Longleftrightarrow\; \ell(\alpha) \in \mathrm{ran}(\ell \circ \tilde\beta) \\
&\Longleftrightarrow\; \ell(\alpha) \in \ell(\beta). \quad \square
\end{aligned}$$

Lemma 3.8 ($\mathbf{CZF} + \mathbf{\Pi\Sigma W{-}AC}$). *Let $\alpha, \beta \in \mathbf{V}(\mathbf{Y}^*_{\mathbf{w}})$. Then we have*

(i) $\exists i \in (\alpha \doteq \beta) \;\Leftrightarrow\; \ell_{\mathbf{w}}(\alpha) = \ell_{\mathbf{w}}(\beta)$.
(ii) $\exists i \in (\alpha \mathrel{\dot\in} \beta) \;\Leftrightarrow\; \ell_{\mathbf{w}}(\alpha) \in \ell_{\mathbf{w}}(\beta)$.

Proof. The same as for Lemma 3.7. □

Definition 3.9 (**CZF**). For any set A and class B we define:

$$A \to B \quad \text{as} \quad \textstyle\prod_{x\in A} B. \tag{10}$$

For any classes A and B we define:

$$A \times B \quad \text{as} \sum_{x \in A} B,$$
$$A + B \quad \text{as} \sum_{x \in 2} C_x, \quad \text{where } C_0 = A \text{ and } C_1 = B. \tag{11}$$

Definition 3.10. A $\mathbf{V}(\mathbf{Y}^*)$-assignment is a mapping $\mathcal{M} : \mathrm{Var} \to \mathbf{V}(\mathbf{Y}^*)$, where Var is the set of variables of the language. $\mathcal{M}(a)$ will also be denoted by $a_{\mathcal{M}}$.

If $\mathcal{M}$ is a $\mathbf{V}(\mathbf{Y}^*)$-assignment, u is a variable, and $d \in \mathbf{V}(\mathbf{Y}^*)$, we define a $\mathbf{V}(\mathbf{Y}^*)$-assignment $\mathcal{M}(u|d)$ by

$$\mathcal{M}(u|d)(v) = \begin{cases} \mathcal{M}(v) & \text{if } v \text{ is a variable other than } u \\ d & \text{if } v \text{ is } u. \end{cases}$$

Sometimes, when an assignment $\mathcal{M}$ is fixed, we will omit the subscript $_{\mathcal{M}}$.

Definition 3.11 (**CZF**). To any bounded formula $\theta \in \mathcal{L}_\in$ and $\mathbf{V}(\mathbf{Y}^*)$-assignment $\mathcal{M}$ we shall assign a set $[\![\theta]\!]_{\mathcal{M}}$. Since we have already used the symbol "$\to$" for function spaces we shall denote the conditional by "$\supset$".

The recursive definition of $[\![\theta]\!]_{\mathcal{M}}$ is given in the table below:

$\theta \in \mathcal{L}_\in$	$[\![\theta]\!]_{\mathcal{M}}$
$\bot$	0
$a = b$	$a_{\mathcal{M}} \doteq b_{\mathcal{M}}$
$a \in b$	$a_{\mathcal{M}} \dot{\in} b_{\mathcal{M}}$
$\theta_0 \wedge \theta_1$	$[\![\theta_0]\!]_{\mathcal{M}} \times [\![\theta_1]\!]_{\mathcal{M}}$
$\theta_0 \vee \theta_1$	$[\![\theta_0]\!]_{\mathcal{M}} + [\![\theta_1]\!]_{\mathcal{M}}$
$\theta_0 \supset \theta_1$	$[\![\theta_0]\!]_{\mathcal{M}} \to [\![\theta_1]\!]_{\mathcal{M}}$
$\forall v \in a\, \psi$	$\prod_{x \in \overline{a_{\mathcal{M}}}} [\![\psi]\!]_{\mathcal{M}(v\|\widetilde{a_{\mathcal{M}}}(x))}$
$\exists v \in a\, \psi$	$\sum_{x \in \overline{a_{\mathcal{M}}}} [\![\psi]\!]_{\mathcal{M}(v\|\widetilde{a_{\mathcal{M}}}(x))}$

Lemma 3.12 (**CZF**). *For every bounded $\theta \in \mathcal{L}_\in$ and $\mathbf{V}(\mathbf{Y}^*)$-assignment $\mathcal{M}$, $[\![\theta]\!]_{\mathcal{M}} \in \mathbf{Y}^*$.*

Proof. This is proved by induction on θ using Lemma 2.8 and Definitions 3.2 and 3.9. $\square$

Lemma 3.13 (**CZF + REA**). *A $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}})$-assignment is defined similarly as a $\mathbf{V}(\mathbf{Y}^*)$-assignment in Definition 3.10. Likewise, as in Definition 3.11, to any bounded formula $\theta \in \mathcal{L}_\in$ and $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}})$-assignment $\mathcal{M}$ we assign a set $[\![\theta]\!]_{\mathcal{M}}$. We then have, for every bounded $\theta \in \mathcal{L}_\in$ and $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}})$-assignment $\mathcal{M}$, $[\![\theta]\!]_{\mathcal{M}} \in \mathbf{Y}^*_{\mathbf{w}}$.*

Proof. This is proved as Lemma 3.12. $\square$

Theorem 3.14 (**CZF + $\Pi\Sigma-$AC**). *For every bounded $\theta \in \mathcal{L}_\in$ and $\mathbf{V}(\mathbf{Y}^*)$-assignment $\mathcal{M}$,*

$$\exists i \in [\![\theta]\!]_{\mathcal{M}} \Leftrightarrow \theta^{\ell(\mathcal{M})},$$

where $\theta^{\ell(\mathcal{M})}$ denotes the result of replacing every free variable a of θ by $\ell(a_{\mathcal{M}})$.

Proof. We proceed by induction on θ. If θ is $\bot$, the assertion is obvious. If θ is $a = b$ or $a \in b$, the assertion follows from Lemma 3.7. Assume θ is $\theta_0 \wedge \theta_1$. Then:

$$\exists i \in [\![\theta_0 \wedge \theta_1]\!]_{\mathcal{M}} \iff \exists u \in [\![\theta_0]\!]_{\mathcal{M}} \wedge \exists v \in [\![\theta_1]\!]_{\mathcal{M}} \overset{\text{IH}}{\iff} \theta_0^{\ell(\mathcal{M})} \wedge \theta_1^{\ell(\mathcal{M})}.$$

Assume θ is $\theta_0 \vee \theta_1$. Then:

$$\begin{aligned}
& \exists i \in [\![\theta_0 \vee \theta_1]\!]_{\mathcal{M}} \\
\Longleftrightarrow\ & \exists u \in 2\, \exists v \big[(u = 0 \wedge v \in [\![\theta_0]\!]_{\mathcal{M}}) \vee (u = 1 \wedge v \in [\![\theta_1]\!]_{\mathcal{M}})\big] \\
\Longleftrightarrow\ & \exists u\, [u = 0\ \wedge\ \exists v \in [\![\theta_0]\!]_{\mathcal{M}}] \vee \exists u\, [u = 1\ \wedge\ \exists v \in [\![\theta_1]\!]_{\mathcal{M}}] \\
\Longleftrightarrow\ & \exists v \in [\![\theta_0]\!]_{\mathcal{M}} \vee \exists v \in [\![\theta_1]\!]_{\mathcal{M}} \\
\overset{\mathrm{IH}}{\Longleftrightarrow}\ & \theta_0^{\ell(\mathcal{M})} \vee \theta_1^{\ell(\mathcal{M})}.
\end{aligned}$$

Assume θ is $(\theta_0 \supset \theta_1)$. Then:

$$\begin{aligned}
& \exists f \in [\![\theta_0 \to \theta_1]\!]_{\mathcal{M}} \\
\Longleftrightarrow\ & \exists f \in ([\![\theta_0]\!]_{\mathcal{M}} \to [\![\theta_1]\!]_{\mathcal{M}}) \\
\Longleftrightarrow\ & \exists f\, (\mathrm{Fun}[f] \wedge \mathrm{dom}(f) = [\![\theta_0]\!]_{\mathcal{M}} \wedge \forall y \in [\![\theta_0]\!]_{\mathcal{M}} (f(y) \in [\![\theta_1]\!]_{\mathcal{M}})) \\
\overset{\mathbf{\Pi\Sigma I-AC}}{\Longleftrightarrow}\ & \forall y \in [\![\theta_0]\!]_{\mathcal{M}} \exists i \in [\![\theta_1]\!]_{\mathcal{M}} \\
\Longleftrightarrow\ & \exists y \in [\![\theta_0]\!]_{\mathcal{M}} \supset \exists i \in [\![\theta_1]\!]_{\mathcal{M}} \\
\overset{\mathrm{IH}}{\Longleftrightarrow}\ & \theta_0^{\ell(\mathcal{M})} \supset \theta_1^{\ell(\mathcal{M})}.
\end{aligned}$$

Assume θ is $\forall v \in a\, \psi$. Then:

$$\begin{aligned}
& \exists f \in [\![\forall v \in a\, \psi]\!]_{\mathcal{M}} \\
\Longleftrightarrow\ & \exists f \in \prod_{x \in \overline{a_{\mathcal{M}}}} [\![\psi]\!]_{\mathcal{M}(v|\widetilde{a_{\mathcal{M}}}(x))} \\
\Longleftrightarrow\ & \exists f\, \big(\mathrm{Fun}(f)\ \wedge\ \mathrm{dom}(f) = \overline{a_{\mathcal{M}}}\ \wedge\ \forall x \in \overline{a_{\mathcal{M}}}\ (f(x) \in [\![\psi]\!]_{\mathcal{M}(v|\widetilde{a_{\mathcal{M}}}(x))})\big) \\
\overset{\mathbf{\Pi\Sigma I-AC}}{\Longleftrightarrow}\ & \forall x \in \overline{a_{\mathcal{M}}}\ \exists i \in [\![\psi]\!]_{\mathcal{M}(v|\widetilde{a_{\mathcal{M}}}(x))} \\
\overset{\mathrm{IH}}{\Longleftrightarrow}\ & \forall x \in \overline{a_{\mathcal{M}}}\ \psi^{\ell(\mathcal{M}(v|\widetilde{a_{\mathcal{M}}}(x))} \\
\Longleftrightarrow\ & (\forall v \in a\, \psi)^{\ell(\mathcal{M})}.
\end{aligned}$$

Assume θ is $\exists v \in a\, \psi$. Then:

$$\begin{aligned}
\exists d \in [\![\exists v \in a\, \psi]\!]_{\mathcal{M}} \Longleftrightarrow\ & \exists d \in \sum_{x \in \overline{a_{\mathcal{M}}}} [\![\psi]\!]_{\mathcal{M}(v|\widetilde{a_{\mathcal{M}}}(x))} \\
\Longleftrightarrow\ & \exists x \in \overline{a_{\mathcal{M}}}\ \exists s \in [\![\psi]\!]_{\mathcal{M}(v|\widetilde{a_{\mathcal{M}}}(x))} \\
\overset{\mathrm{IH}}{\Longleftrightarrow}\ & \exists x \in \overline{a_{\mathcal{M}}}\ \psi^{\ell(\mathcal{M}(v|\widetilde{a_{\mathcal{M}}}(x))} \\
\Longleftrightarrow\ & (\exists v \in a\, \psi)^{\ell(\mathcal{M})}. \quad \square
\end{aligned}$$

Theorem 3.15 (**CZF** + **REA** + $\mathbf{\Pi\Sigma W-AC}$). *For every bounded* $\theta \in \mathcal{L}_\in$ *and* $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}})$*-assignment* $\mathcal{M}$*,*

$$\exists i \in [\![\theta]\!]_{\mathcal{M}} \ \Leftrightarrow\ \theta^{\ell_{\mathbf{w}}(\mathcal{M})},$$

where $\theta^{\ell_{\mathbf{w}}(\mathcal{M})}$ *denotes the result of replacing every free variable* a *of* θ *by* $\ell_{\mathbf{w}}(a_{\mathcal{M}})$.

Proof. is by induction on θ as in the previous Theorem 3.14. $\square$

4. The formulae-as-classes interpretation for arbitrary formulae

In order to reflect within **CZF** the formulae-as-classes interpretation for arbitrary set-theoretic formulae and judgements of Martin-Löf type theory we shall need to represent large types $\Pi\Sigma$-generated on top of $\mathbf{V}(\mathbf{Y}^*)$. The language of **CZF**, though, is not rich enough to do it in a straightforward way. To remedy this we utilize a special notion of set recursive partial function developed in [21].

4.1. Extended E-recursive functions

We would like to have unlimited application of sets to sets, i.e. we would like to assign a meaning to the symbol $\{a\}(x)$ where a and x are sets. In generalized recursion theory this is known as *E-recursion* or *set recursion* (see, e.g., [18] or [23, Ch.X]). However, we shall introduce an extended notion of E-computability, christened $E_\wp$-computability, rendering the function $exp(a, b) = {}^a b$ is computable as well (where ${}^a b$ denotes the set of all functions from a to b). Moreover, the constant function with value ω is taken as an initial function in $E_\wp$-computability. From a classical standpoint, $E_\wp$-computability is related to power recursion, where the power set operation is regarded to be an initial function. The latter notion has been studied by Moschovakis [15] and Moss [16].

There is a lot of leeway in setting up $E_\wp$-recursion. The particular schemes we use are especially germane to our situation. Our construction will provide a specific set-theoretic model for the elementary theory of operations and numbers **EON** (see, e.g., [7, VI.2], or the theory **APP** as described in [24, Ch. 9, Sect. 3]).

Definition 4.1 (**CZF**). We shall utilize encoding of finite sequences of sets by the pairing function $\langle\, ,\rangle$. First, we select distinct non-zero natural numbers $\mathbf{k}$, $\mathbf{s}$, $\mathbf{p}$, $\mathbf{p_0}$, $\mathbf{p_1}$, $\mathbf{s_N}$, $\mathbf{p_N}$, $\mathbf{d_N}$, $\bar{\mathbf{0}}$, $\bar{\omega}$, $\boldsymbol{\pi}$, $\boldsymbol{\sigma}$, $\overline{\boldsymbol{pl}}$, $\boldsymbol{i}$, $\overline{\mathbf{fa}}$, and $\overline{\mathbf{ab}}$ which will provide indices for special $E_\wp$-recursive partial (class) functions. Inductively we shall define a class $\mathbb{E}$ of triples $\langle e, x, y\rangle$. Rather than "$\langle e, x, y\rangle \in \mathbb{E}$", we shall write "$\{e\}(x) \simeq y$", and moreover, if $n > 0$, we shall use $\{e\}(x_1, \ldots, x_n) \simeq y$ to convey that $\{e\}(x_1) \simeq \langle e, x_1\rangle$ and $\{\langle e, x_1\rangle\}(x_2) \simeq \langle e, x_1, x_2\rangle$ and ... and $\{\langle e, x_1, \ldots, x_{n-1}\rangle\}(x_n) \simeq y$. We shall say that $\{e\}(x)$ is defined, written $\{e\}(x)\downarrow$, if $\{e\}(x) \simeq y$ for some y. Let $\mathbb{N} := \omega$. $\mathbb{E}$ is defined by the following clauses (inference steps):

$$
\begin{array}{rcl}
\{\mathbf{k}\}(x, y) & \simeq & x \\
\{\mathbf{s}\}(x, y, z) & \simeq & \{\{x\}(z)\}(\{y\}(z)) \\
\{\mathbf{p}\}(x, y) & \simeq & \langle x, y\rangle \\
\{\mathbf{p_0}\}(x) & \simeq & (x)_0 \\
\{\mathbf{p_1}\}(x) & \simeq & (x)_1 \\
\{\mathbf{s_N}\}(n) & \simeq & n + 1 \text{ if } n \in \mathbb{N} \\
\{\mathbf{p_N}\}(0) & \simeq & 0 \\
\{\mathbf{p_N}\}(n + 1) & \simeq & n \text{ if } n \in \mathbb{N} \\
\{\mathbf{d_N}\}(n, m, x, y) & \simeq & x \text{ if } n, m \in \mathbb{N} \text{ and } n = m \\
\{\mathbf{d_N}\}(n, m, x, y) & \simeq & y \text{ if } n, m \in \mathbb{N} \text{ and } n \neq m \\
\{\bar{\mathbf{0}}\}(x) & \simeq & 0 \\
\{\bar{\omega}\}(x) & \simeq & \omega \\
\{\boldsymbol{\pi}\}(x, g) & \simeq & \prod_{z\in x} g(z) \text{ if } g \text{ is a (set-)function with } \mathrm{dom}(g) = x \\
\{\boldsymbol{\sigma}\}(x, g) & \simeq & \sum_{z\in x} g(z) \text{ if } g \text{ is a (set-)function with } \mathrm{dom}(g) = x \\
\{\overline{\boldsymbol{pl}}\}(x, y) & \simeq & x + y \\
\{\boldsymbol{i}\}(x, y, z) & \simeq & \mathbf{I}(x, y, z) \\
\{\overline{\mathbf{fa}}\}(g, x) & \simeq & g(x) \text{ if } g \text{ is a (set-)function and } x \in \mathrm{dom}(g) \\
\{\overline{\mathbf{ab}}\}(e, a) & \simeq & h \text{ if } h \text{ is a (set-)function with } \mathrm{dom}(h) = a \\
 & & \quad\text{and } \forall x{\in}a\ \{e\}(x) \simeq h(x).
\end{array}
$$

Note that for $\{\mathbf{s}\}(x, y, z)$ to be defined it is required that $\{x\}(z)$, $\{y\}(z)$ and $\{\{x\}(z)\}(\{y\}(z))$ be defined. The clause for $\mathbf{s}$ is thus to be read as a conjunction of the following clauses: $\{\mathbf{s}\}(x) \simeq \langle \mathbf{s}, x\rangle$, $\{\langle \mathbf{s}, x\rangle\}(y) \simeq \langle \mathbf{s}, x, y\rangle$ and, if there exist a, b, c such that $\{x\}(z) \simeq a$, $\{y\}(z) \simeq b$, $\{a\}(b) \simeq c$, then $\{\langle \mathbf{s}, x, y\rangle\}(z) \simeq c$.

The constants $\overline{\mathbf{fa}}$ and $\overline{\mathbf{ab}}$ stand for *function application* and *function abstraction*, respectively.

Lemma 4.2 (**CZF**). *$\mathbb{E}$ is an inductively defined class and $\mathbb{E}$ is functional in that for all e, x, y, y',*

$$\langle e, x, y\rangle \in \mathbb{E} \ \wedge\ \langle e, x, y'\rangle \in \mathbb{E} \ \Rightarrow\ y = y'.$$

Proof. The inductive definition of $\mathbb{E}$ falls under the heading of Lemma 2.7. If $\{e\}(x) \simeq y$ the uniqueness of y follows by induction on the stages (see Lemma 2.7) of that inductive definition. $\square$

Definition 4.3. *Application terms* are defined inductively as follows:

(i) The constants $\mathbf{k}$, $\mathbf{s}$, $\mathbf{p}$, $\mathbf{p_0}$, $\mathbf{p_1}$, $\mathbf{s_N}$, $\mathbf{p_N}$, $\mathbf{d_N}$, $\bar{\mathbf{0}}$, $\bar{\omega}$, $\boldsymbol{\pi}$, $\boldsymbol{\sigma}$, $\overline{\boldsymbol{pl}}$, $\boldsymbol{i}$, $\overline{\mathbf{fa}}$, $\overline{\mathbf{ab}}$ singled out in Definition 4.1 are *application terms*;
(ii) Variables are *application terms*;
(iii) If s and t are *application terms* then (st) is an *application term*.

Definition 4.4. Application terms are easily formalized in **CZF**. However, rather than translating application terms into the set-theoretic language of **CZF**, we translate expressions of the form $t \simeq u$ into the language of set theory, where t is an application term and u is a variable.

The translation proceeds along the build-up of t as follows:

$$\begin{array}{rll} c \simeq u & \text{is} & c = u \text{ if } c \text{ is a constant or a variable;} \\ (st) \simeq u & \text{is} & \exists x \exists y (s \simeq x \wedge t \simeq y \wedge \{x\}(y) \simeq u). \end{array}$$

Abbreviations. In connection with application terms $s, t, t_1, \ldots, t_n$ we will use the following abbreviations:

$$\begin{array}{rll} s(t_1, \ldots, t_n) & \text{is short for} & ((\ldots(st_1)\ldots)t_n) \quad \textit{(parentheses associated to the left)}; \\ st_1 \ldots t_n & \text{is short for} & s(t_1, \ldots, t_n); \\ t\downarrow & \text{is short for} & \exists x\ t \simeq x; \quad (t \textit{ is defined}) \\ s \simeq t & \text{is short for} & s\downarrow \vee t\downarrow \ \supset \ \exists x (s \simeq x \wedge t \simeq x). \end{array}$$

A *closed* application term is an application term that does not contain variables. If t is a closed application term and $a_1, \ldots, a_n, b$ are sets we use the abbreviation

$$t(a_1, \ldots, a_n) \simeq b \quad \text{ for } \exists x_1 \ldots x_n \exists y \big(x_1 = a_1 \wedge \ldots \wedge x_n = a_n \wedge y = b \\ \wedge\ t(x_1, \ldots, x_n) \simeq y\big).$$

Definition 4.5. Every closed application term gives rise to a partial class function. A partial n-place (class) function Υ is said to be an *$E_\wp$-recursive partial function* if there exists a closed application term t_Υ such that

$$\mathrm{dom}(\Upsilon) = \{(a_1, \ldots, a_n) \mid t_\Upsilon(a_1, \ldots, a_n)\downarrow\}$$

and for all for all sets $(a_1, \ldots, a_n) \in \mathrm{dom}(\Upsilon)$,

$$t_\Upsilon(a_1, \ldots, a_n) \simeq \Upsilon(a_1, \ldots, a_n).$$

In the latter case, t_Υ is said to be an *index* for Υ.

If Υ_1, Υ_2 are $E_\wp$-recursive partial functions, then $\Upsilon_1(\vec{a}) \simeq \Upsilon_2(\vec{a})$ iff neither $\Upsilon_1(\vec{a})$ nor $\Upsilon_2(\vec{a})$ are defined, or $\Upsilon_1(\vec{a})$ and $\Upsilon_2(\vec{a})$ are defined and equal.

The next two results can be proved in the theory **APP** and thus hold true in any applicative structure. Thence the above applicative structure satisfies the Abstraction Lemma and Recursion Theorem (see e.g. [10] or [7]).

Lemma 4.6 (*Abstraction Lemma, cf. [7, VI.2.2]*). *For every application term t there exists an application term $\lambda x.t$ whose free variables are those of t without x such that the following holds:*

$$\forall x_1 \ldots \forall x_n (\lambda x.t\downarrow \ \wedge\ \forall y\, (\lambda x.t)y \simeq t[x/y]).$$

Proof. (i) $\lambda x.x$ is $\mathbf{skk}$; (ii) $\lambda x.t$ is $\mathbf{k}t$ for t a constant or a variable other than x; (iii) $\lambda x.uv$ is $(\mathbf{s}(\lambda x.u))(\lambda x.v)$. □

Lemma 4.7 (*Recursion Theorem, cf. [7, VI.2.7]*). *There exists a closed application term* rec *such that for any* f, x,

$$\mathsf{rec} f\downarrow \ \wedge\ \mathsf{rec} f x \simeq f(\mathsf{rec} f)x.$$

Proof. Take rec to be $\lambda f.tt$, where t is $\lambda y \lambda x. f(yy)x$. □

Corollary 4.8. *For any $E_\wp$-recursive partial function Υ there exists a closed application term τ_{fix} such that $\tau_{fix}\downarrow$ and for all $\vec{a}$,*

$$\Upsilon(\bar{e}, \vec{a}) \simeq \tau_{fix}(\vec{a}),$$

where $\tau_{fix} \simeq \bar{e}$. Moreover, τ_{fix} can be effectively (e.g. primitive recursively) constructed from an index for Υ.

4.2. Arbitrary formulae

With the aid of indices for $E_\wp$-recursive partial functions, we have the means to extend the formulae-as-sets interpretation of Definition 3.11 to arbitrary formulae. However, while $[\![\theta]\!]_\mathcal{M}$ is a set for bounded formulae θ, the interpretation $[\![\varphi]\!]^{\cdot}_\mathcal{M}$ of an unbounded formula will be a proper class (where $\mathcal{M}$ is a $\mathbf{V}(\mathbf{Y}^*)$-assignment). For example, if φ is of the form $\varphi_0 \supset \varphi_1$ with φ_0 unbounded, then $[\![\varphi]\!]^{\cdot}_\mathcal{M}$ will be a class of indices of $E_\wp$-recursive partial functions that map elements of $[\![\varphi_0]\!]^{\cdot}_\mathcal{M}$ to elements of $[\![\varphi_1]\!]^{\cdot}_\mathcal{M}$.

In the course of defining $[\![\varphi]\!]^{\cdot}_\mathcal{M}$, we shall also furnish this class with an equality relation. As in Martin-Löf type theory, each class $[\![\varphi]\!]^{\cdot}_\mathcal{M}$ comes equipped with its own equality relation, and we shall write $a = b \in [\![\varphi]\!]^{\cdot}_\mathcal{M}$ to convey that a and b are "equal" elements of $[\![\varphi]\!]^{\cdot}_\mathcal{M}$. For bounded φ, $a = b \in [\![\varphi]\!]^{\cdot}_\mathcal{M}$ just means the ordinary identity of a and b as sets, i.e., $a = b \in [\![\varphi]\!]^{\cdot}_\mathcal{M}$ iff $a, b \in [\![\varphi]\!]^{\cdot}_\mathcal{M}$ and $a = b$. For formulae θ, ψ with θ unbounded we define

$$[\![\theta]\!]^{\cdot}_\mathcal{M} \xrightarrow{\sim} [\![\psi]\!]^{\cdot}_\mathcal{M} \tag{12}$$

to be the class of all sets s such that for all $x \in [\![\theta]\!]^{\cdot}_\mathcal{M}$, $\{s\}(x) \in [\![\psi]\!]^{\cdot}_\mathcal{M}$ and for all $x = y \in [\![\theta]\!]^{\cdot}_\mathcal{M}$, $\{s\}(x) = \{s\}(y) \in [\![\psi]\!]^{\cdot}_\mathcal{M}$, assuming that the equality relations on $[\![\theta]\!]^{\cdot}_\mathcal{M}$ and $[\![\psi]\!]^{\cdot}_\mathcal{M}$ have been previously defined.

Definition 4.9 (**CZF**). If B is a class and a, x are sets, we write $\{a\}(x) \mathrel{\underset{\sim}{\in}} B$ for $\exists y(\{a\}(x) \simeq y \wedge y \in B)$.

If A is a class and B_x are classes for all $x \in A$, then we define a class $\widetilde{\prod}_{x\in A} B_x$ in the following way:

$$\widetilde{\prod}_{x\in A} B_x := \{a \mid \forall x \in A(\{a\}(x) \in B_x)\}. \tag{13}$$

Definition 4.10 (**CZF**). For every formula $\theta \in \mathcal{L}_\in$ and $\mathbf{V}(\mathbf{Y}^*)$-assignment $\mathcal{M}$, we define a class $[\![\theta]\!]^{\cdot}_\mathcal{M}$. The definition is given in the table below:

$\theta \in \mathcal{L}_\in$	$[\![\theta]\!]^{\cdot}_\mathcal{M}$
$\bot$	0
$a = b$	$a_\mathcal{M} \doteq b_\mathcal{M}$
$a \in b$	$a_\mathcal{M} \mathrel{\dot\in} b_\mathcal{M}$
$\theta_0 \wedge \theta_1$	$[\![\theta_0]\!]^{\cdot}_\mathcal{M} \times [\![\theta_1]\!]^{\cdot}_\mathcal{M}$
$\theta_0 \vee \theta_1$	$[\![\theta_0]\!]^{\cdot}_\mathcal{M} + [\![\theta_1]\!]^{\cdot}_\mathcal{M}$
$\theta_0 \supset \theta_1$	$[\![\theta_0]\!]_\mathcal{M} \to [\![\theta_1]\!]^{\cdot}_\mathcal{M}$ if θ_0 is bounded
$\theta_0 \supset \theta_1$	$[\![\theta_0]\!]^{\cdot}_\mathcal{M} \xrightarrow{\sim} [\![\theta_1]\!]^{\cdot}_\mathcal{M}$ if θ_0 is not bounded
$\forall v \in a\, \psi$	$\prod_{x\in\overline{a_\mathcal{M}}} [\![\psi]\!]^{\cdot}_{\mathcal{M}(v\mid\widetilde{a_\mathcal{M}}(x))}$
$\exists v \in a\, \psi$	$\sum_{x\in\overline{a_\mathcal{M}}} [\![\psi]\!]^{\cdot}_{\mathcal{M}(v\mid\widetilde{a_\mathcal{M}}(x))}$
$\forall v\psi$	$\widetilde{\prod}_{x\in\mathbf{V}(\mathbf{Y}^*)} [\![\psi]\!]^{\cdot}_{\mathcal{M}(v\mid x)}$
$\exists v\psi$	$\sum_{x\in\mathbf{V}(\mathbf{Y}^*)} [\![\psi]\!]^{\cdot}_{\mathcal{M}(v\mid x)}$

We also have to declare the equality relations pertaining to the above classes. For bounded θ, $x = y \in [\![\theta]\!]^{\cdot}_\mathcal{M}$ stands for $x, y \in [\![\theta]\!]^{\cdot}_\mathcal{M}$ and $x = y$. $\langle x, y\rangle = \langle u, v\rangle \in [\![\theta_0]\!]^{\cdot}_\mathcal{M} \times [\![\theta_1]\!]^{\cdot}_\mathcal{M}$ means $\langle x, y\rangle, \langle u, v\rangle \in [\![\theta_0]\!]^{\cdot}_\mathcal{M} \times [\![\theta_1]\!]^{\cdot}_\mathcal{M}$ and $x = u \in [\![\theta_0]\!]^{\cdot}_\mathcal{M}$ and $y = v \in [\![\theta_1]\!]^{\cdot}_\mathcal{M}$. $\langle i, x\rangle = \langle j, y\rangle \in [\![\theta_0]\!]^{\cdot}_\mathcal{M} + [\![\theta_1]\!]^{\cdot}_\mathcal{M}$ means $\langle i, x\rangle, \langle j, y\rangle \in [\![\theta_0]\!]^{\cdot}_\mathcal{M} + [\![\theta_1]\!]^{\cdot}_\mathcal{M}$ and either $i = j = 0 \wedge x = y \in [\![\theta_0]\!]^{\cdot}_\mathcal{M}$ or $i = j = 1 \wedge x = y \in [\![\theta_1]\!]^{\cdot}_\mathcal{M}$. For bounded θ_0, $f = g \in \big([\![\theta_0]\!]_\mathcal{M} \to [\![\theta_1]\!]^{\cdot}_\mathcal{M}\big)$ means $f, g \in \big([\![\theta_0]\!]_\mathcal{M} \to [\![\theta_1]\!]^{\cdot}_\mathcal{M}\big)$ and $f = g$. For unbounded θ_0, $a = b \in \big([\![\theta_0]\!]_\mathcal{M} \xrightarrow{\sim} [\![\theta_1]\!]^{\cdot}_\mathcal{M}\big)$ means $a, b \in \big([\![\theta_0]\!]_\mathcal{M} \xrightarrow{\sim} [\![\theta_1]\!]^{\cdot}_\mathcal{M}\big)$ and for all $x \in [\![\theta_0]\!]_\mathcal{M}$, $\{a\}(x) = \{b\}(x) \in [\![\theta_1]\!]_\mathcal{M}$. $f = g \in \prod_{x\in\overline{a_\mathcal{M}}} [\![\psi]\!]^{\cdot}_{\mathcal{M}(v\mid\widetilde{a_\mathcal{M}}(x))}$ means $f, g \in \prod_{x\in\overline{a_\mathcal{M}}} [\![\psi]\!]^{\cdot}_{\mathcal{M}(v\mid\widetilde{a_\mathcal{M}}(x))}$ and for all $x \in \overline{a_\mathcal{M}}$, $f(x) = g(x) \in [\![\psi]\!]^{\cdot}_{\mathcal{M}(v\mid\widetilde{a_\mathcal{M}}(x))}$. $\langle y, z\rangle = \langle u, w\rangle \in \sum_{x\in\overline{a_\mathcal{M}}} [\![\psi]\!]^{\cdot}_{\mathcal{M}(v\mid\widetilde{a_\mathcal{M}}(x))}$ means $\langle y, z\rangle, \langle u, w\rangle \in \sum_{x\in\overline{a_\mathcal{M}}} [\![\psi]\!]^{\cdot}_{\mathcal{M}(v\mid\widetilde{a_\mathcal{M}}(x))}$ and $y = z$ and $u = w \in [\![\psi]\!]^{\cdot}_{\mathcal{M}(v\mid\widetilde{a_\mathcal{M}}(y))}$. $a = b \in \widetilde{\prod}_{x\in\mathbf{V}(\mathbf{Y}^*)} [\![\psi]\!]^{\cdot}_{\mathcal{M}(v\mid x)}$ means $a, b \in \widetilde{\prod}_{x\in\mathbf{V}(\mathbf{Y}^*)} [\![\psi]\!]^{\cdot}_{\mathcal{M}(v\mid x)}$ and for all $x \in \mathbf{V}(\mathbf{Y}^*)$, $\{a\}(x) = \{b\}(x) \in$

$[\![\psi]\!]^{\cdot}_{\mathcal{M}(v|x)}$. $\langle y,z\rangle = \langle u,w\rangle \in \sum_{x\in \mathbf{V}(\mathbf{Y}^*)}[\![\psi]\!]^{\cdot}_{\mathcal{M}(v|x)}$ means $\langle y,z\rangle, \langle u,w\rangle \in \sum_{x\in \mathbf{V}(\mathbf{Y}^*)}[\![\psi]\!]^{\cdot}_{\mathcal{M}(v|x)}$ and $y = u$ and $z = w \in [\![\psi]\!]^{\cdot}_{\mathcal{M}(v|y)}$.

Lemma 4.11 (**CZF**). *For every bounded formula* θ *and* $\mathbf{V}(\mathbf{Y}^*)$*-assignment* $\mathcal{M}$, $[\![\theta]\!]_{\mathcal{M}} = [\![\theta]\!]^{\cdot}_{\mathcal{M}}$.

Proof. This follows by induction on θ by comparing 3.11 and 4.10. □

Definition 4.12. If $\theta(u_1,\ldots,u_r)$ is a formula of $\mathcal{L}_\in$ all of whose free variables are among $u_1,\ldots,u_r$, and $\alpha_1,\ldots,\alpha_r \in \mathbf{V}(\mathbf{Y}^*)$, we shall use the shorthand $[\![\theta(\alpha_1,\ldots,\alpha_r)]\!]$ rather than $[\![\theta]\!]^{\cdot}_{\mathcal{M}}$ whenever $\mathcal{M}$ is an assignment satisfying $\mathcal{M}(u_i) = \alpha_i$ for $1 \le i \le r$. In the special case when θ is a sentence we will simply write $[\![\theta]\!]$. We shall also use the following abbreviations:

$$\begin{array}{lll} e \Vdash \theta(\alpha_1,\ldots,\alpha_r) & \text{iff} & e \in [\![\theta(\alpha_1,\ldots,\alpha_r)]\!] \\ \mathbf{V}(\mathbf{Y}^*) \models \theta(\alpha_1,\ldots,\alpha_r) & \text{iff} & e \Vdash \theta(\alpha_1,\ldots,\alpha_r) \text{ for some } e \\ \models^* \theta(\alpha_1,\ldots,\alpha_r) & \text{iff} & \mathbf{V}(\mathbf{Y}^*) \models \theta(\alpha_1,\ldots,\alpha_r). \end{array}$$

For a set-theoretic formula $\theta(\vec{u})$ we say that $\theta(\vec{\alpha})$ is *validated in* $\mathbf{V}(\mathbf{Y}^*)$ if we have produced a closed application term t such that $t(\vec{\alpha}) \Vdash \theta(\vec{\alpha})$ holds for all $\vec{\alpha} \in \mathbf{V}(\mathbf{Y}^*)$.

4.3. *The formulae-as-classes interpretation for* **CZF**

The rationale for the employment of the particular notion of extended E-recursive is revealed only in the proof of the following theorem.

Theorem 4.13 (*[21], Theorem 4.13*). *Let* $\theta(u_1,\ldots,u_r)$ *be a formula of* $\mathcal{L}_\in$ *all of whose free variables are among* $u_1,\ldots,u_r$*. If*

$$\mathbf{CZF} + \mathbf{\Pi\Sigma{-}AC} \vdash \theta(u_1,\ldots,u_r),$$

then one can effectively construct an index of a $E_\wp$*-recursive partial function* g *such that*

$$\mathbf{CZF}_{Exp} \vdash \forall \alpha_1,\ldots,\alpha_r \in \mathbf{V}(\mathbf{Y}^*)\ \ g(\alpha_1,\ldots,\alpha_r) \in [\![\theta(\alpha_1,\ldots,\alpha_r)]\!].$$

Recall that $\mathbf{CZF}_{Exp}$ *denotes the modification of* **CZF** *with Exponentiation in place of Subset Collection.*

Proof. See [21], Theorem 4.13. The proof of 4.13 is rather long and requires close attention to the definition of indices of $E_\wp$-recursive functions. □

4.4. *The formulae-as-classes interpretation for* **CZF + REA**

As the reader might expect, the formulae-as-classes interpretation given for **CZF** above can be extended to **CZF + REA** also. The first step is to add the following condition to the definition of $E_\wp$-recursive functions, giving rise to the $E^w_\wp$*-recursive functions*:

$$\{\bar{\mathbf{w}}\}(x,g) = \mathbf{W}_{z\in x} g(z) \text{ if } g \text{ is a (set-)function with } \mathrm{dom}(g) = x,$$

where $\bar{\mathbf{w}}$ is a "fresh" natural number.

One then defines for every formula $\theta \in \mathcal{L}_\in$ and $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}})$-assignment $\mathcal{M}$, a class $[\![\theta]\!]^{\cdot}_{\mathcal{M}}$ as in Definition 4.10, where, however, the definition of the product

$$\widetilde{\prod}_{x\in A} B_x := \{a \mid \forall x \in A(\{a\}(x) \in B_x)\} \tag{14}$$

is to be understood in the sense of $E^w_\wp$-recursive functions. Correspondingly, we obtain the following result.

Theorem 4.14 (*[21], Theorem 4.33*). *Let* $\theta(u_1,\ldots,u_r)$ *be a formula of* $\mathcal{L}_\in$ *all of whose free variables are among* $u_1,\ldots,u_r$*. If*

$$\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W{-}AC} \vdash \theta(u_1,\ldots,u_r),$$

then one can effectively construct an index of a $E^{w}_{\wp}$-recursive partial function g such that

$$\mathbf{CZF}_{Exp} + \mathbf{REA} \vdash \forall \vec{\alpha} \in \mathbf{V}(\mathbf{Y}^*)\ \ g(\vec{\alpha}) \in [\![\theta(\vec{\alpha})]\!],$$

where $\vec{\alpha} = \alpha_1, \ldots, \alpha_r$ and $\mathbf{CZF}_{Exp}$ denotes the modification of **CZF** *with Exponentiation in place of Subset Collection.*

Proof. See [21], Theorem 4.33. The proof builds on the proof of Theorem 4.13. $\square$

5. The formulae-as-classes interpretation and validity in $\mathbf{H}(\mathbf{Y}^*)$

The following considerations are reminiscent of Definition 3.8 and Theorem 3 of [25].

Definition 5.1. A formula is said to be CC if no unbounded quantifier in it occurs in the antecedent of an implication.

Note that bounded as well as prenex (i.e. bounded preceded by a string of quantifiers) formulae are CC.

Theorem 5.2 ($\mathbf{CZF} + \mathbf{\Pi\Sigma{-}AC}$). *For every $\theta \in \mathcal{L}_\in$ and any $\mathbf{V}(\mathbf{Y}^*)$-assignment $\mathcal{M}$, if θ is CC, then*

$$\exists i \in [\![\theta]\!]^{\cdot}_{\mathcal{M}} \Rightarrow \theta^{\ell(\mathcal{M})},$$

where $\theta^{\ell(\mathcal{M})}$ denotes the result of replacing each free variable a of θ by $\ell(a_{\mathcal{M}})$ and each unbounded quantifier Qx of θ by $Qx \in \mathbf{H}(\mathbf{Y}^*)$.

The **proof** is by induction on θ. If θ is an atom, the assertion follows from Lemma 4.11 and Theorem 3.14. If θ is a conjunction or disjunction, then the assertion follows easily from the IH.

Suppose θ is $\theta_0 \supset \theta_1$ and $\mathfrak{x} \in [\![\theta]\!]^{\cdot}_{\mathcal{M}}$. Since $\theta \in \mathsf{CC}$, θ_0 must be bounded. If $\theta_0^{\ell(\mathcal{M})}$, then by Theorem 3.14 $\exists i \in [\![\theta_0]\!]_{\mathcal{M}}$, and thus $\mathfrak{x}(i) \in [\![\theta_1]\!]^{\cdot}_{\mathcal{M}}$, which by the IH yields $\theta_1^{\ell(\mathcal{M})}$. As a result, $\theta^{\ell(\mathcal{M})}$.

Assume θ is $\forall v \in a\, \psi$. Then we have:

$$\begin{aligned}
& \exists f \in [\![\forall v \in a\, \psi]\!]^{\cdot}_{\mathcal{M}} \\
\Longleftrightarrow\ & \exists f \in \prod_{x \in \overline{a_{\mathcal{M}}}} [\![\psi]\!]^{\cdot}_{\mathcal{M}(v|\widetilde{a_{\mathcal{M}}}(x))} \\
\Longleftrightarrow\ & \exists f \left[\mathrm{Fun}[f] \ \wedge\ \mathrm{dom}(f) = \overline{a_{\mathcal{M}}} \ \wedge\ \forall x \in \overline{a_{\mathcal{M}}}\big(f(x) \in [\![\psi]\!]^{\cdot}_{\mathcal{M}(v|\widetilde{a_{\mathcal{M}}}(x))}\big)\right] \\
\Longrightarrow\ & \forall x \in \overline{a_{\mathcal{M}}}\, \exists y \in [\![\psi]\!]^{\cdot}_{\mathcal{M}(v|\widetilde{a_{\mathcal{M}}}(x))} \\
\overset{\mathrm{IH}}{\Longrightarrow}\ & \forall x \in \overline{a_{\mathcal{M}}}\, \psi^{\ell(\mathcal{M}(v|\widetilde{a_{\mathcal{M}}}(x)))} \\
\Longleftrightarrow\ & (\forall v \in a\, \psi)^{\ell(\mathcal{M})}.
\end{aligned}$$

Assume θ is $\exists v \in a\, \psi$. Then:

$$\begin{aligned}
\exists d \in [\![\exists v \in a\, \psi]\!]^{\cdot}_{\mathcal{M}} \Longleftrightarrow\ & \exists d \in \sum_{j \in \overline{a_{\mathcal{M}}}} [\![\psi]\!]^{\cdot}_{\mathcal{M}(v|j)} \\
\Longleftrightarrow\ & \exists j \in \overline{a_{\mathcal{M}}}\, \exists s \in [\![\psi]\!]^{\cdot}_{\mathcal{M}(v|j)} \\
\overset{\mathrm{IH}}{\Longrightarrow}\ & \exists j \in \overline{a_{\mathcal{M}}}\ \psi^{\ell(\mathcal{M}(v|\widetilde{a_{\mathcal{M}}}(j))} \\
\Longleftrightarrow\ & (\exists v \in a\, \psi)^{\ell(\mathcal{M})}.
\end{aligned}$$

Assume θ is $\forall v\, \psi$. Then we have:

$$\begin{aligned}
\exists a \in [\![\forall v\, \psi]\!]^{\cdot}_{\mathcal{M}} \Longleftrightarrow\ & \exists a \in \widetilde{\prod}_{x \in \mathbf{V}(\mathbf{Y}^*)} [\![\psi]\!]^{\cdot}_{\mathcal{M}(v|x)} \\
\Longleftrightarrow\ & \exists a\, \forall x \in \mathbf{V}(\mathbf{Y}^*) \left(\{a\}(x) \in [\![\psi]\!]^{\cdot}_{\mathcal{M}(v|x)}\right) \\
\Longrightarrow\ & \forall x \in \mathbf{V}(\mathbf{Y}^*)\ \exists y \in [\![\psi]\!]^{\cdot}_{\mathcal{M}(v|x)} \\
\overset{\mathrm{IH}}{\Longrightarrow}\ & \forall x \in \mathbf{V}(\mathbf{Y}^*)\ \psi^{\ell(\mathcal{M}(v|x))} \\
\overset{\mathrm{L.3.6}}{\Longrightarrow}\ & (\forall v\, \psi)^{\ell(\mathcal{M})}.
\end{aligned}$$

Assume θ is $\exists v\, \psi$. Then:

$$\begin{aligned}
\exists d \in [\![\exists v\, \psi]\!]^{\cdot}_{\mathcal{M}} &\iff \exists d \in \sum_{j\in\mathbf{V}(\mathbf{Y}^*)} [\![\psi]\!]^{\cdot}_{\mathcal{M}(v|j)} \\
&\iff \exists j \in \mathbf{V}(\mathbf{Y}^*)\ \exists s \in [\![\psi]\!]^{\cdot}_{\mathcal{M}(v|j)} \\
&\stackrel{\mathrm{IH}}{\Longrightarrow} \exists j \in \mathbf{V}(\mathbf{Y}^*) \psi^{\ell(\mathcal{M}(v|j))} \\
&\Longrightarrow (\exists v\, \psi)^{\ell(\mathcal{M})}. \quad \square
\end{aligned}$$

Theorem 5.3 (**CZF** + **REA** + **ΠΣW−AC**). *For every $\theta \in \mathcal{L}_\in$ and any $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}})$-assignment $\mathcal{M}$, if θ is* CC, *then*

$$\exists i \in [\![\theta]\!]^{\cdot}_{\mathcal{M}} \Rightarrow \theta^{\ell_{\mathbf{w}}(\mathcal{M})},$$

where $\theta^{\ell_{\mathbf{w}}(\mathcal{M})}$ denotes the result of replacing each free variable a of θ by $\ell_{\mathbf{w}}(a_{\mathcal{M}})$ and each unbounded quantifier Qx of θ by $Qx \in \mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$.

Proof. This is the same proof as for the previous one. $\square$

5.1. "Mathematical" formulae

The previous theorem provides a collection of formulae for which inhabitedness of their formulae-as-classes interpretation implies their truth. However, it is not clear whether this collection includes many statements of workaday mathematics. To show the richness of CC, we shall coin the notion of a "mathematical" formula.

Definition 5.4. The *mathematical set terms* are a collection of class terms inductively defined by the following clauses:

1. ω is a mathematical set term.
2. If S and T are mathematical set terms then so are
$$\begin{aligned}\bigcup S &:= \{u : \exists x \in S\ u \in x\},\\ \{S, T\} &:= \{u : u = S \vee u = T\}.\end{aligned}$$
3. If S and T are mathematical set terms then so are
$$\begin{aligned}S + T &:= \{\langle 0, x\rangle : x \in S\} \cup \{\langle 1, x\rangle : x \in T\},\\ S \times T &:= \{\langle x, y\rangle : x \in S \wedge y \in T\},\\ S \to T &:= \{f : f : S \to T\}.\end{aligned}$$
4. If $S, T_1, \ldots, T_n$ are mathematical set terms and $\psi(x, y_1, \ldots, y_n)$ is a restricted formula (of set theory) then
$$\{x \in S : \psi(x, T_1, \ldots, T_n)\}$$
is a mathematical set term.
5. If $S, T_1, \ldots, T_n, P_1, \ldots, P_k$ are mathematical set terms and $\psi(x, \vec{y}, \vec{z})$ is a bounded formula (of set theory), where $\vec{y}, \vec{z} = y_1, \ldots, y_n, z_1, \ldots, z_k$, then
$$\{u : u = \{x \in S : \psi(x, y_1, \ldots, y_n, \vec{P})\} \wedge y_1 \in T_1 \wedge \ldots \wedge y_n \in T_n)\}$$
is a mathematical set term, where $\vec{P} = P_1, \ldots, P_k$.

The *generalized mathematical set terms* are defined by the clauses for mathematical set terms plus the following clauses:

6. If T is a generalized mathematical set term then so is $\mathbf{H}(T)$, where $\mathbf{H}(T)$ denotes the smallest class Y such that $\mathrm{ran}(f) \in Y$ whenever $a \in T$ and $f : a \to Y$.
7. If S and T are generalized mathematical set terms, then so is $\mathbf{W}_{x\in S} T_x$.
8. If S and T are generalized mathematical set terms, then so is $\mathbf{WF}(S, T)$.
   Here $\mathbf{WF}(S, T)$ denotes the smallest class Z such that whenever $a\in S$ and $T_a = \{x\in S \mid \langle x, a\rangle \in T\} \subseteq Z$ then $a\in Z$.

A *mathematical formula* (*generalized mathematical formula*) is a formula of the form $\psi(T_1, \ldots, T_n)$, where $\psi(x_1, \ldots, x_n)$ is bounded and $T_1, \ldots, T_n$ are mathematical set terms (generalized mathematical set terms) (with the proviso that none of the free variables occurring in the T_i's is a bound variable of ψ).

A *mathematical sentence* (*generalized mathematical sentence*) is a mathematical formula (generalized mathematical formula) without free variables.

Remark 5.5.

1. From the point of view of **ZFC**, the mathematical set terms denote sets of rank $< \omega+\omega$ in the cumulative hierarchy while the generalized mathematical set terms denote sets of rank $< \aleph_\omega$.
2. The idea behind mathematical set terms is that they comprise all sets that one is interested in in ordinary mathematics. E.g., with the help of Definition 5.4, clauses (1) and (3) one constructs the set of natural numbers, integers, rationals, and the function space $\mathbb{N} \to \mathbb{Q}$. Using clause (4) one obtains the set of Cauchy sequences of rationals from $\mathbb{N} \to \mathbb{Q}$. The main application of clause (5) is made in constructing quotients. If S and $R \subseteq S \times S$ are set terms and R is an equivalence relation on S, then (5) permits one to form the set term

$$S/R \ = \{[a]_R \mid a \in S\},$$

where $[a]_R = \{x \in S \mid \langle x, a\rangle \in R\}$.

Therefore, by employing clause (5), one can define the set of equivalence classes of Cauchy sequences, i.e., the set of reals.
3. Definition 5.4 clause (5) is related to the abstraction axiom of Friedman's system **B** in [11].

Lemma 5.6. (i) (**CZF**) *Every mathematical set term is a set.*
(ii) (**CZF** + **REA**) *Every generalized mathematical set term is a set.*

Proof. We proceed by induction on the clauses for the definition of mathematical set terms. ω is a set by the Infinity Axiom. That the set terms generated by clause (2) are sets follows from the respective inductive hypothesis via the Pairing and Union Axioms. If the set terms are generated according to clause (3), one applies the respective inductive hypothesis and the fact that **CZF** proves the existence of the disjoint union, cartesian product, and function space of any two sets. For set terms generated according to clause (4) one uses the inductive hypothesis for the set terms $S, T_1, \ldots, T_n$ and Bounded Separation. Next, we address clause (5). By the inductive hypotheses, $\vec{P}, \vec{T}, S$ are sets. Hence, using Bounded Separation, $\{x \in S : \ \psi(x, \vec{y}, \vec{P})\}$ is a set for every $\vec{y} \in T_1 \times \cdots \times T_n$. Using the Replacement Schema (which is provable in **CZF**),

$$\{u : \ u = \{x \in S : \ \psi(x, y_1, \ldots, y_n, \vec{P})\} \wedge y_1 \in T_1 \wedge \ldots \wedge y_n \in T_n)\}$$

is a set.

To prove that every generalized mathematical set term is a set on the basis of **CZF** + **REA**, we have to consider clauses (6)–(8) as well. Here we invoke [3], Corollary 5.3, namely that **CZF** + **REA** proves that $\mathbf{H}(T)$, $\mathbf{W}_{x\in S}T_x$ and $\mathbf{WF}(S, T)$ are sets whenever S and T are sets. $\Box$

Formally, we shall conceive of mathematical formulae and generalized mathematical formulae as defined in a certain extension $\mathcal{L}_{class}$ of the language $\mathcal{L}_\in$, namely an extension by class terms. Strictly speaking, the formulae-as-classes interpretation is defined for formulae of $\mathcal{L}_\in$ only. In order to talk about the interpretation of $\mathcal{L}_\in$-formulae, we shall fix a translation $(\cdot)^\diamond$ from $\mathcal{L}_{class}$ to $\mathcal{L}_\in$. The definition below is inductive and follows the intended meaning of generalized mathematical set terms in Definition 5.4.

Definition 5.7. We first define $(x = S)^\diamond$ for set terms S by recursion on the build-up of S:

$$\begin{aligned}
(x = \omega)^\diamond &:= \forall u\big[u \in x \leftrightarrow \big(0 = u \ \vee \ \exists v \in x(u = v \cup \{v\})\big)\big] \\
\Big(x = \bigcup S\Big)^\diamond &:= \exists z\left[(z = S)^\diamond \ \wedge \ \forall u(u \in x \leftrightarrow \exists v \in z\ u \in v)\right] \\
(x = \{S, T\})^\diamond &:= \exists z \exists w\left[(z = S)^\diamond \ \wedge \ (w = T)^\diamond \ \wedge \right. \\
&\qquad \left. \forall u(u \in x \leftrightarrow u = z \vee u = w)\right] \\
(x = S + T)^\diamond &:= \exists z \exists w\left[(z = S)^\diamond \ \wedge \ (w = T)^\diamond \ \wedge \right.
\end{aligned}$$

$$\begin{aligned}
&\qquad\qquad\qquad\forall u(u \in x \leftrightarrow \exists v \in z\, u = \langle 0, v\rangle \vee \exists y \in w\, u = \langle 1, y\rangle)\big] \\
(x = S \times T)^{\diamondsuit} &:= \exists z \exists w \left[(z = S)^{\diamondsuit} \ \wedge\ (w = T)^{\diamondsuit} \ \wedge \right. \\
&\qquad \forall u(u \in x \leftrightarrow \exists x \in z \exists y \in w\, u = \langle x, y\rangle)\big] \\
(x = S \to T)^{\diamondsuit} &:= \exists z \exists w \left[(z = S)^{\diamondsuit} \ \wedge\ (w = T)^{\diamondsuit} \ \wedge \right. \\
&\qquad \forall f [f \in x \leftrightarrow \mathrm{Fun}(f) \wedge \mathrm{dom}(f) = z \wedge \forall y \in z (f(y) \in w)]\big].
\end{aligned}$$

If Q is a set term of the form $\{v \in S :\ \psi(v, \vec{T})\}$ then[1]

$$\begin{aligned}
(x = Q)^{\diamondsuit} := \exists z \exists \vec{w} &\left[(z = S)^{\diamondsuit} \ \wedge\ \left(\vec{w} = \vec{T}\right)^{\diamondsuit} \ \wedge \right. \\
&\forall v(v \in x \leftrightarrow v \in z \wedge \psi(v, \vec{w}))\big].
\end{aligned}$$

If Q is a set term of the form $\{u :\ u = \{v \in S :\ \psi(v, \vec{y}, \vec{P})\} \ \wedge\ \vec{y} \in \vec{T}\}$, then $(x = Q)^{\diamondsuit}$ is the formula

$$\begin{aligned}
&\exists z \exists \vec{w}\, \exists \vec{y} \left[(z = S)^{\diamondsuit} \ \wedge\ \left(\vec{w} = \vec{T}\right)^{\diamondsuit} \ \wedge\ \left(\vec{y} = \vec{P}\right)^{\diamondsuit} \ \wedge \right. \\
&\forall u(u \in x \leftrightarrow \exists \vec{v} \in \vec{w}\, u = \{p \in z :\ \psi(p, \vec{v}, \vec{y})\})\big],
\end{aligned}$$

where $u = \{p \in z :\ \psi(p, \vec{v}, \vec{y})\}$ stands for

$$\forall q \in u[q \in z \wedge \psi(q, \vec{v}, \vec{y})] \wedge \forall q \in z[\psi(q, \vec{v}, \vec{y}) \to q \in u].$$

In the case of generalized mathematical set terms we have to consider three more cases.

Suppose Q is of the form $\mathbf{H}(T)$, where T is a generalized mathematical set term. Put

$$\begin{aligned}
\psi_{\mathbf{H}}(a, b) \ &:=\ \forall f\, \forall u \in a\big[\mathrm{Fun}(f) \ \wedge\ \mathrm{dom}(f) = u \ \wedge\ \mathrm{ran}(f) \subseteq b \ \to \\
&\qquad\qquad \exists z \in b\, [z = \mathrm{ran}(f)]\big], \\
(x = \mathbf{H}(T))^{\diamondsuit} &:= \exists z \left[(z = T)^{\diamondsuit} \ \wedge\ \psi_{\mathbf{H}}(z, x) \ \wedge \right. \\
&\qquad\qquad \forall w\, [\psi_{\mathbf{H}}(z, w) \to x \subseteq w]\big].
\end{aligned}$$

Suppose Q is of the form $\mathbf{W}_{x \in S} T_x$, where S and T are generalized mathematical set terms. Put

$$\begin{aligned}
\psi_{\mathbf{W}}(a, b, c) &:= \forall f\, \forall u \in a\big[\mathrm{Fun}(f) \ \wedge\ \mathrm{dom}(f) = b_u \ \wedge\ \mathrm{ran}(f) \subseteq c \ \to \\
&\qquad\qquad \langle u,\ f\rangle \in c\big], \\
(x = \mathbf{W}_{u \in S} T_u)^{\diamondsuit} &:= \exists z\, \exists v \left[(z = S)^{\diamondsuit} \ \wedge\ (v = T)^{\diamondsuit} \ \wedge\ \psi_{\mathbf{W}}(z, v, x) \ \wedge \right. \\
&\qquad\qquad \forall w\, [\psi_{\mathbf{W}}(z, v, w) \to x \subseteq w]\big].
\end{aligned}$$

Suppose Q is of the form $\mathbf{WF}(S, R)$, where S and R are generalized mathematical set terms. Put

$$\begin{aligned}
&\psi_{\mathbf{WF}}(a, r, c) := \forall u \in a\big[\forall v(\langle v, u\rangle \in r \to v \in c) \ \to\ u \in c\big], \\
&(x = \mathbf{WF}(S, R))^{\diamondsuit} := \exists z\, \exists r \left[(z = S)^{\diamondsuit} \ \wedge\ (r = R)^{\diamondsuit} \ \wedge\ \psi_{\mathbf{WF}}(z, r, x) \ \wedge \right. \\
&\qquad\qquad\qquad \forall w\, [\psi_{\mathbf{WF}}(z, r, w) \to x \subseteq w]\big].
\end{aligned}$$

An arbitrary mathematical formula (generalized mathematical formula) is of the form $\psi(T_1, \ldots, T_n)$, where $T_1, \ldots, T_n$ are mathematical set terms (generalized mathematical set terms) and $\psi(z_1, \ldots, z_n)$ is a bounded formula of $\mathcal{L}_{\in}$. We then put

$$(\psi(T_1, \ldots, T_n))^{\diamondsuit} \ := \exists z_1 \ldots \exists z_n \left[\left(\vec{z} = \vec{T}\right)^{\diamondsuit} \wedge \psi(z_1, \ldots, z_n)\right].$$

The reason for bothering the reader with a detailed translation of $\mathcal{L}_{class}$-formulae into the official language of set theory is that an inspection of it readily yields the following result.

[1] For a vector of set terms $\vec{T} \equiv T_1, \ldots, T_n$ we write $\vec{y} \in \vec{T}$ and $\left(\vec{y} = \vec{T}\right)^{\diamondsuit}$ for $y_1 \in T_1 \ \wedge \ldots \wedge\ y_n \in T_n$ and $(y_1 \in T_1)^{\diamondsuit} \ \wedge \ldots \wedge\ (y_n \in T_n)^{\diamondsuit}$, respectively.

Lemma 5.8. *If θ is a mathematical formula then $\theta^{\diamond}$ belongs to the* CC *formulae.*

This leads to the following corollaries of Theorems 5.2 and 5.3.

Theorem 5.9 $(\mathbf{CZF} + \mathbf{\Pi\Sigma{-}AC})$. *For every mathematical formula θ and $\mathbf{V}(\mathbf{Y}^*)$-assignment $\mathcal{M}$,*

$$\exists i \in [\![\theta^{\diamond}]\!]^{\cdot}_{\mathcal{M}} \quad \textit{implies} \quad \left(\theta^{\diamond}\right)^{\ell(\mathcal{M})},$$

where for a formula ψ, $\psi^{\ell(\mathcal{M})}$ denotes the result of replacing each free variable a of ψ by $\ell(a_{\mathcal{M}})$ and each unbounded quantifier Qx of ψ by $Qx \in \mathbf{H}(\mathbf{Y}^*)$.

Theorem 5.10 $(\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W{-}AC})$. *For every mathematical formula θ and a $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}})$-assignment $\mathcal{M}$*

$$\exists i \in [\![\theta^{\diamond}]\!]^{\cdot}_{\mathcal{M}} \quad \textit{implies} \quad \left(\theta^{\diamond}\right)^{\ell_{\mathbf{w}}(\mathcal{M})},$$

where for a formula ψ, $\psi^{\ell_{\mathbf{w}}(\mathcal{M})}$ denotes the result of replacing each free variable a of ψ by $\ell_{\mathbf{w}}(a_{\mathcal{M}})$ and each unbounded quantifier Qx of ψ by $Qx \in \mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$.

We would like to expand the previous result to generalized mathematical formulae, the obstacle being that these formulae need not be in CC.

Definition 5.11. A class A is *regular* if it is transitive and for every $a \in A$ and set $R \subseteq a \times A$, if $\forall x \in a\, \exists y\, \langle x, y\rangle \in R$ then there is a set $b \in A$ such that

$$\forall x \in a\, \exists y \in b\, \langle x, y\rangle \in R \ \wedge\ \forall y \in b\, \exists x \in a\, \langle x, y\rangle \in R.$$

Definition 5.12. Let $\mathbf{\Pi\Sigma{-}PAx}$ be the assertion that every $\mathbf{\Pi\Sigma}$-generated set is a base and every set is an image of a $\mathbf{\Pi\Sigma}$-generated set. Similarly, one defines $\mathbf{\Pi\Sigma W{-}PAx}$.

Lemma 5.13 $(\mathbf{CZF} + \mathbf{\Pi\Sigma{-}AC})$. (i) $\mathbf{H}(\mathbf{Y}^*)$ *is a regular model of* $\mathbf{CZF} + \mathbf{DC} + \mathbf{\Pi\Sigma{-}AC} + \mathbf{\Pi\Sigma{-}PAx}$.
$(\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W{-}AC})$ (ii) $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$ *is a regular model of* $\mathbf{CZF} + \mathbf{REA} + \mathbf{DC} + \mathbf{\Pi\Sigma{-}AC} + \mathbf{\Pi\Sigma W{-}PAx}$.

Proof. By Lemma 3.4, we have $\mathbf{H}(\mathbf{Y}) = \mathbf{H}(\mathbf{Y}^*)$ and $\mathbf{H}(\mathbf{Y}_{\mathbf{w}}) = \mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$. (1) then follows from [3], Theorem 4.2 and (2) follows from [3], Theorem 5.10. □

Definition 5.14. If θ is a generalized mathematical formula with parameters in $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}})$ we shall use the abbreviation

$$\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}}) \models \theta \quad := \quad \exists i \in [\![\theta^{\diamond}]\!]_{\mathcal{M}}.$$

Likewise, if θ is a generalized mathematical formula with parameters in $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$ we shall use the abbreviation

$$\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}}) \models \theta$$

iff $\theta^{\diamond}$ holds in $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$, i.e. with all unbounded quantifiers restricted to $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$.

Lemma 5.15 $(\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W{-}AC})$. *Let $\alpha, \beta, \gamma \in \mathbf{V}(\mathbf{Y}^*_{\mathbf{w}})$ and $\dot{\alpha} = \ell_{\mathbf{w}}(\alpha)$, $\dot{\beta} = \ell_{\mathbf{w}}(\beta)$, and $\dot{\gamma} = \ell_{\mathbf{w}}(\gamma)$. Then we have the following.*

$$\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}}) \models \beta = \mathbf{H}(\alpha) \quad \Rightarrow \quad \mathbf{H}(\mathbf{Y}^*_{\mathbf{w}}) \models \dot{\beta} = \mathbf{H}(\dot{\alpha}). \tag{15}$$

$$\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}}) \models \gamma = \mathbf{W}_{u\in\alpha}\beta_u \quad \Rightarrow \quad \mathbf{H}(\mathbf{Y}^*_{\mathbf{w}}) \models \dot{\gamma} = \mathbf{W}_{u\in\dot{\alpha}}\dot{\beta}_u. \tag{16}$$

$$\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}}) \models \gamma = \mathbf{WF}(\alpha, \beta) \quad \Rightarrow \quad \mathbf{H}(\mathbf{Y}^*_{\mathbf{w}}) \models \dot{\gamma} = \mathbf{WF}(\dot{\alpha}, \dot{\beta}). \tag{17}$$

Proof. Assume $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}}) \models \beta = \mathbf{H}(\alpha)$. The formula $\psi_{\mathbf{H}}(\alpha, \beta)$ of Definition 5.7 is a formula which starts with a universal quantifier and is followed by a bounded matrix, and thus, by Theorem 5.3,

$$\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}}) \models \psi_{\mathbf{H}}(\dot{\alpha}, \dot{\beta}). \tag{18}$$

Since $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$ is a model of $\mathbf{CZF} + \mathbf{REA}$ by Lemma 5.13, there exists $b \in \mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$ such that $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}}) \models b = \mathbf{H}(\dot{\alpha})$. As $\ell_{\mathbf{w}}$ is surjective there exists $\rho \in \mathbf{V}(\mathbf{Y}^*_{\mathbf{w}})$ such that $\dot{\rho} = b$. From (18) we deduce $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}}) \models \dot{\rho} \subseteq \dot{\beta}$, and hence, using Theorem 3.15,

$$\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}}) \models \rho \subseteq \beta. \tag{19}$$

Next we would like to show that also $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}}) \models \beta \subseteq \rho$. Here we have to resort to a different description of $\mathbf{H}(\alpha)$. By Lemma 2.7, we have that provably in **CZF**,

$$x \in \mathbf{H}(\alpha) \quad \Leftrightarrow \quad \exists G \exists u [G \text{ is good } \wedge \ x \in G^a], \tag{20}$$

where *"G is good"* stands for

$$\forall \langle v, y \rangle \in G \, \exists b \in \alpha \, \exists f \left[\mathrm{Fun}(f) \ \wedge \ f : b \to G^{\in v} \ \wedge \ \mathrm{ran}(f) = y\right]\!.$$

Letting $\psi_g(\alpha, x)$ denote the formula on the right hand side of (20), we see that $\psi_g(\alpha, x)$ belongs to CC.

Now suppose $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}}) \models \eta \in \beta$. As $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}})$ is a model of **CZF** by Theorem 4.14, we can employ the foregoing considerations to express this fact via the CC formula $\psi_g(\alpha, \eta)$, so that $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}}) \models \psi_g(\alpha, \eta)$ and therefore, by Theorem 5.3, $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}}) \models \psi_g(\dot{\alpha}, \dot{\eta})$. As $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$ is a model of **CZF** as well we arrive at $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}}) \models \dot{\eta} \in H(\dot{\alpha})$. Hence $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}}) \models \dot{\eta} \in \dot{\rho}$ and so (by Theorem 3.15) $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}}) \models \eta \in \rho$, showing that $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}}) \models \beta \subseteq \rho$. Thus, in conjunction with (19), we get $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}}) \models \beta = \rho$, yielding

$$\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}}) \models \dot{\beta} = \mathbf{H}(\dot{\alpha}).$$

The proofs of the other cases are similar and utilize the same considerations. $\square$

Theorem 5.16 (**CZF** + **REA** + **ΠΣW−AC**). *For every generalized mathematical formula θ and $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}})$-assignment $\mathcal{M}$,*

$$\exists i \in [\![\theta^{\diamond}]\!]^{\cdot}_{\mathcal{M}} \quad \textit{implies} \quad \left(\theta^{\diamond}\right)^{\ell_{\mathbf{w}}(\mathcal{M})},$$

where for a formula ψ, $\psi^{\ell_{\mathbf{w}}(\mathcal{M})}$ denotes the result of replacing each parameter a of ψ by $\ell_{\mathbf{w}}(a_{\mathcal{M}})$ and each unbounded quantifier Qx of ψ by $Qx \in \mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$.

Proof. $\theta^{\diamond}$ is of the form $\exists z_1 \ldots \exists z_n \left[\left(\vec{z} = \vec{T}\right)^{\diamond} \wedge \psi(z_1, \ldots, z_n)\right]$, where $\psi(\vec{z})$ is a bounded formula and the $\vec{T}$ are generalized set terms. The assertion then follows from Lemma 5.15 taken together with Theorem 3.15. $\square$

5.2. Absoluteness of mathematical formulae

In this subsection we show that mathematical formulae are absolute for $\mathbf{H}(\mathbf{Y}^*)$ and that generalized mathematical formulae are absolute for $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$.

Lemma 5.17 (**CZF** + **ΠΣ−AC**). *Let S be a set term with parameters in $\mathbf{H}(\mathbf{Y}^*)$. By Lemma 5.13, $\mathbf{H}(\mathbf{Y}^*)$ is a model of* **CZF**, *and thus S is interpreted as a set in $\mathbf{H}(\mathbf{Y}^*)$. Let $S^{\mathbf{H}(\mathbf{Y}^*)}$ be the interpretation of S in $\mathbf{H}(\mathbf{Y}^*)$. Then $S = S^{\mathbf{H}(\mathbf{Y}^*)}$.*

Proof. The proof proceeds by induction on the generation of S. Note that except for the case when S is of the form $T \to P$, this is obvious because of the absoluteness of bounded formulae.

Suppose S is of the form $T \to P$. From the inductive hypotheses for T and P we get $T = T^{\mathbf{H}(\mathbf{Y}^*)}$ and $P = P^{\mathbf{H}(\mathbf{Y}^*)}$, in particular $T, P \in \mathbf{H}(\mathbf{Y}^*)$. Since $\mathbf{H}(\mathbf{Y}^*)$ is a model of **CZF** it suffices to show that $(T \to P) \subseteq \mathbf{H}(\mathbf{Y}^*)$ to be able to conclude that $(T \to P) = (T \to P)^{\mathbf{H}(\mathbf{Y}^*)}$. Let $f : T \to P$. Since $T \in \mathbf{H}(\mathbf{Y}^*)$ there exists $A \in \mathbf{Y}^*$ and $g : A \to T$ such that $T = \mathrm{ran}(g)$. Now define $h : A \to \mathbf{H}(\mathbf{Y}^*)$ by

$$h(i) \ = \ \langle g(i), f(g(i)) \rangle.$$

Then $\mathrm{ran}(h) \in \mathbf{H}(\mathbf{Y}^*)$ and, moreover, $\mathrm{ran}(h) = f$, whence $f \in \mathbf{H}(\mathbf{Y}^*)$. $\square$

Lemma 5.18 (**CZF** + **REA** + **ΠΣW−AC**). *Let S be a generalized set term with parameters in $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$. By Lemma 5.13, $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$ is a model of* **CZF** + **REA**, *and thus S is interpreted as a set in $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$. Let $S^{\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})}$ be the interpretation of S in $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$. Then $S = S^{\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})}$.*

Proof. Again, the proof proceeds by induction on the generation of S. In addition to the cases of the previous lemma, we have to consider inductively defined set terms. Suppose $S = \mathbf{H}(T)$. By the inductive assumption we then have $T^{\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})} = T$. We will call a set of ordered pairs G *good* if

$$\forall \langle a, y \rangle \in G \, \exists f \exists b {\in} T \, [\, f : b \to G^{\in a} \ \wedge \ y = \mathrm{ran}(f)],$$

where $G^{\in a} = \bigcup_{b \in a} G^b$ and $G^b = \{u \mid \langle b, u \rangle \in G\}$.

By Lemma 2.7 we get

$$x \in (\mathbf{H}(T))^{\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})} \quad \text{iff} \quad \mathbf{H}(\mathbf{Y}^*_{\mathbf{w}}) \models \exists G\, \exists a\, [G \text{ is good} \wedge x \in G^a].$$

As the property of being good is formalizable by a Σ formula and therefore upward persistent, $x \in (\mathbf{H}(T))^{\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})}$ implies $\exists G\, \exists a\, [G \text{ is good} \wedge x \in G^a]$, and thence, by Lemma 2.7, $x \in \mathbf{H}(T)$. In consequence, $(\mathbf{H}(T))^{\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})} \subseteq \mathbf{H}(T)$. To establish the converse inclusion, suppose $c \in T$ and $f : c \to (\mathbf{H}(T))^{\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})}$. In the course of the proof of Lemma 5.17 it was shown that the latter yields $f \in (\mathbf{H}(T))^{\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})}$, hence we get $\mathrm{ran}(f) \in (\mathbf{H}(T))^{\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})}$. Having shown that $(\mathbf{H}(T))^{\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})}$ is closed under the clauses defining $\mathbf{H}(T)$, we conclude $\mathbf{H}(T) \subseteq (\mathbf{H}(T))^{\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})}$.

The cases where $S = \mathbf{W}_{x \in P} T_x$ or $S = \mathbf{WF}(P, R)$ are dealt with in the same way as in the case of $S = \mathbf{H}(T)$. □

Definition 5.19. Let $\Sigma(math)$ ($\Sigma(gmath)$) denote the smallest collection of formulae which comprises the mathematical set formulae (the generalized mathematical set formulae) and is closed under $\wedge$, $\vee$, bounded quantification, and unbounded existential quantification.

Corollary 5.20. ($\mathbf{CZF} + \mathbf{\Pi\Sigma}-\mathbf{AC}$) (i) *Let ψ be a $\Sigma(math)$ formula with parameters in $\mathbf{H}(\mathbf{Y}^*)$. If $\mathbf{H}(\mathbf{Y}^*) \models \psi$, then ψ.*
($\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma}-\mathbf{AC}$) (ii) *Let ψ be a $\Sigma(gmath)$ formula with parameters in $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}})$. If $\mathbf{H}(\mathbf{Y}^*_{\mathbf{w}}) \models \psi$, then ψ.*

Proof. This follows readily by induction on ψ using Lemma 5.17 and Lemma 5.18, respectively. □

Theorem 5.21. ($\mathbf{CZF} + \mathbf{\Pi\Sigma}-\mathbf{AC}$) (i) *Let θ be a $\Sigma(math)$ sentence. If $\mathbf{V}(\mathbf{Y}^*) \models \psi$, then ψ holds true.*
($\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W}-\mathbf{AC}$) (ii) *Let θ be a $\Sigma(gmath)$ sentence. If $\mathbf{V}(\mathbf{Y}^*_{\mathbf{w}}) \models \psi$, then ψ holds true.*

Proof. (i) is a consequence of Theorem 5.9 and Corollary 5.20, (i), while (ii) follows from Theorem 5.16 in conjunction with Corollary 5.20, (ii). □

[2,3] feature several more choice principles. The main reason for their omission is that these axioms have no impact on the preceding result. This will be made precise below.

Definition 5.22. Let $\mathbf{BCA}_\Pi$ be the statement that whenever A is a base and B_a is a base for each $a \in A$, then $\prod_{x \in A} B_x$ is a base.

Let $\mathbf{BCA}_\mathrm{I}$ be the statement that whenever A is a base then $\mathbf{I}(A, b, c)$ is a base for all $b, c \in A$.

Theorem 5.23. *Let ψ be a mathematical sentence and let θ be a generalized mathematical sentence. Then the following hold:*

(i) $\mathbf{CZF} + \mathbf{\Pi\Sigma}-\mathbf{AC} \vdash \psi$ *if and only if*
$\mathbf{CZF} + \mathbf{\Pi\Sigma}-\mathbf{AC} + \mathbf{\Pi\Sigma}-\mathbf{PAx} + \mathbf{BCA}_\Pi + \mathbf{BCA}_\mathrm{I} + \mathbf{RDC} \vdash \psi$.
(ii) $\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W}-\mathbf{AC} \vdash \theta$ *if and only if*
$\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W}-\mathbf{PAx} + \mathbf{BCA}_\Pi + \mathbf{BCA}_\mathrm{I} + \mathbf{RDC} \vdash \theta$.

Proof. (i): Arguing in $\mathbf{CZF}+\mathbf{\Pi\Sigma}-\mathbf{AC}$ one can show that $\mathbf{H}(\mathbf{Y}^*)$ is a model of $\mathbf{CZF}+\mathbf{\Pi\Sigma}-\mathbf{AC}+\mathbf{RDC}+\mathbf{\Pi\Sigma}-\mathbf{AC}+\mathbf{\Pi\Sigma}-\mathbf{PAx}$ by the same proof as for [3], Theorem 4.2. By Corollary 2.12, $\mathbf{\Pi\Sigma}-\mathbf{AC}$ and $\mathbf{\Pi\Sigma I}-\mathbf{AC}$ are equivalent over $\mathbf{CZF}$, and $\mathbf{\Pi\Sigma I}-\mathbf{AC}$ implies $\mathbf{BCA}_\Pi$ and $\mathbf{BCA}_\mathrm{I}$. To see this note that by 4.8 of [2] the class of bases is the class of those sets that are in one–one correspondence with a $\mathbf{\Pi\Sigma I}$-generated set from which it follows that the class of bases is $\mathbf{\Pi\Sigma I}$-closed and hence $\mathbf{BCA}_\Pi$ and $\mathbf{BCA}_\mathrm{I}$ hold. The upshot is that $\mathbf{H}(\mathbf{Y}^*)$ is also a model of $\mathbf{BCA}_\Pi$ and $\mathbf{BCA}_\mathrm{I}$. Hence (i) follows owing to Corollary 5.20(i).
(ii) is proved similarly, this time by utilizing Corollary 5.20(ii) and [3] Theorem 5.10. □

6. Interpretations of type theory in CZF and CZF + REA

In the series of papers [1–3], Aczel gave interpretations of $\mathbf{CZF}$ and $\mathbf{CZF} + \mathbf{REA}$ in Martin-Löf's intuitionistic type theory. Upon nearer examination, one can delineate respective systems $\mathbf{ML_1V}$ and $\mathbf{ML_1WV}$ of type theory that are sufficient unto these tasks of interpretation. In what follows we assume familiarity with type theory as presented in Martin-Löf's 1984 monograph [14] or in Beeson's book [7]. It is perhaps worth pointing out that the treatment of equality in this version of type theory differs from the earlier one in [13]. The type theory of [14] has also been called *extensional type theory*. In [13] there is one relation of definitional equality which is engendered by the principles that a definiendum is always definitionally equal to its definiens and that definitional equality is preserved under substitutions. In the version [14] of type theory that we are concerned with, each type is equipped with its own equality relation which is not necessarily to be understood as a definitional equality. For each type A, the expression $a = b : A$ is used to convey the judgement that a and b are equal elements of type A. Above all, it is to be observed that the equality $f = g : A \to B$ of the type of functions from A to B means that f and g are extensionally equal, i.e. $f(x) = g(x) : A \to B$.

The basic system of type theory, notated by $\mathbf{ML_0}$, is the one with the type constructors $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{N}_1$, Π, Σ, $+$, $\mathbf{I}$. In [13,14] Martin-Löf considered an infinite, externally indexed tower of universes $\mathbf{U}_1 \in \mathbf{U}_2 \in \ldots \in \mathbf{U}_n \in \ldots$ all of which are closed under the standard ensemble of type forming operations. By $\mathbf{ML_1}$ we shall denote the extension of $\mathbf{ML_0}$ by one universe $\mathbf{U}$ plus rules to the effect that $\mathbf{U}$ is closed under the above constructors. $\mathbf{ML_{1W}}$ denotes the extension of $\mathbf{ML_1}$ wherein the universe $\mathbf{U}$ is also closed under taking $\mathbf{W}$-types (see 6.1 below). The formalization of universes for intuitionistic type theory we use in this section is that referred to as the *Russell formulation* in Martin-Löf's monograph [14]. The fundamental notions of type theory are introduced in the four forms of judgement: *A is a type* (abbr. *A type*), *A and B are equal types* (abbr. $A = B$), *a is an element of type A* (abbr. $a : A$), and *a, b are equal elements of type A* (abbr. $a = b : A$). We prefer to use the colon ":" rather than the elementhood symbol "∈" to stress the distinction between set theory and type theory. The rule of type theory are presented in natural deduction style as in [14]. The judgements within brackets indicate discharged assumptions.

Definition 6.1. The introduction rules of $\mathbf{ML_{1W}}$ concerning the $\mathbf{W}$-type are the following:

$$\frac{A : \mathbf{U} \quad \begin{matrix}[x : A] \\ F(x) : \mathbf{U}\end{matrix}}{\mathbf{W}(A, F) : \mathbf{U}}$$

$$\frac{\mathbf{W}(A, F) : \mathbf{U}}{\mathbf{W}(A, F) \text{ type.}}$$

Combining the foregoing rules gives rise to the derived rule of restricted $\mathbf{W}$-formation,

$$(\text{res-}\mathbf{W}\text{-formation}) \qquad \frac{A : \mathbf{U} \quad \begin{matrix}[x : A] \\ F(x) : \mathbf{U}\end{matrix}}{\mathbf{W}(A, F) \text{ type}}.$$

Definition 6.2. The theories $\mathbf{ML_1V}$ and $\mathbf{ML_{1W}V}$ are obtained from $\mathbf{ML_1}$ and $\mathbf{ML_{1W}}$, respectively, by equipping them with Aczel's type of iterative sets $\mathbf{V}$ (cf. [1]). The rules pertaining to $\mathbf{V}$ are:

$$(\mathbf{V}\text{-formation}) \qquad \mathbf{V} \text{ type}$$

$$(\mathbf{V}\text{-introduction}) \qquad \frac{A : \mathbf{U} \quad f : A \to \mathbf{V}}{\mathbf{sup}(A, f) : \mathbf{V}}$$

$$(\mathbf{V}\text{-elimination}) \qquad \frac{c : \mathbf{V} \quad \begin{matrix}[A : \mathbf{U}, f : A \to \mathbf{V}] \\ [z : (\mathbf{\Pi} v : A) C(f(v))] \\ d(A, f, z) : C(\mathbf{sup}(A, f))\end{matrix}}{\mathbf{T_V}(c, (A, f, z)d) : C(c)}$$

$$(\mathbf{V}\text{-equality}) \qquad \frac{B : \mathbf{U} \quad g : B \to \mathbf{V} \quad \begin{matrix}[A : \mathbf{U}, f : A \to \mathbf{V}] \\ [z : (\mathbf{\Pi} v : A) C(f(v))] \\ d(A, f, z) : C(\mathbf{sup}(A, f))\end{matrix}}{\mathbf{T_V}(\mathbf{sup}(B, g), (A, f, z)d) = \mathbf{t}_{B,g,A,f,z,d} : C(\mathbf{sup}(B, g))},$$

where $\mathbf{t}_{B,g,A,f,z,d} := d(B, g, (\boldsymbol{\lambda} v)\mathbf{T_V}(g(v), (A, f, z)d))$.

In order to define their interpretations in set theory, we need a detailed account of the syntax of $\mathbf{ML_1V}$ and $\mathbf{ML_{1W}V}$. Here we will follow [7, Ch. XI]; however, for the readers' convenience, we shall recall most of the definitions. If B is any expression, and $x_1, \ldots, x_n$ are variables, we form the expression $(x_1, \ldots, x_n)B$. The symbol $\stackrel{\triangle}{\equiv}$ will be used for the relation on expressions satisfying

$$((x_1, \ldots, x_n)B)(x_1, \ldots, x_n) \stackrel{\triangle}{\equiv} B$$

and $A \stackrel{\triangle}{\equiv} C$ for expressions A and C which differ only in the renaming of bound variables (cf. [7, XI6]).

Definition 6.3 (*cf. [7, XI.20.3]*). The *constants* of $\mathbf{ML_1V}$ are: $\mathbf{\Pi}$, $\mathbf{\Sigma}$, $\mathbf{I}$, $\mathbf{+}$, $\mathbf{N}$, $\mathbf{0}$, $\mathbf{s_N}$, $\mathbf{r}$, $\boldsymbol{\lambda}$, $\mathbf{ap}$, $\mathbf{sup}$, $\mathbf{E}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{D}$, $\mathbf{J}$, $\mathbf{R}$, $\mathbf{T_V}$, $\mathbf{U}$, $\mathbf{V}$ and for each natural number m, $\mathbf{N_m}$ and $\mathbf{R_m}$. $\mathbf{ML_{1W}V}$ also has the constant $\mathbf{W}$. The *terms* are generated by:

1. Every constant and variable is a term;
2. If t and s are terms, then $t(s)$ and (t, s) are terms;
3. If t is a term, then $(x_1, \ldots, x_n)t$ is a term, where the x_i are variables.

Free and bound occurrences of variables in terms are defined as usual, letting abstraction, i.e. the formation of $(x_1, \ldots, x_n)t$, bind the variables $x_1, \ldots, x_n$. We now would like to assign to every term t of $\mathbf{ML_1V}$ a corresponding application term t^* by replacing the abstract application of $\mathbf{ML_1V}$ with set-recursive application. It is then a straightforward matter to translate a formula of the form $t^* : X$ into a legitimate formula of $\mathbf{CZF}$.

Definition 6.4. We now assign to each term t of $\mathbf{ML_1V}$ an application term t^*. Occurrences of λ in the definition of t^* denote the λ-operator introduced by Lemma 4.6. We fix two new natural numbers $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{v}}$. We shall write (x, y) for $\mathbf{p}(x, y)$ and, inductively, $(x_1, \ldots, x_{k+1})$ for $\mathbf{p}((x_1, \ldots, x_k), x_{k+1})$. For constants $\mathbf{c}$ we define $\mathbf{c}^*$ by:

$$\begin{aligned}
\mathbf{0}^* &\text{ is } 0\\
\mathbf{\Pi}^* &\text{ is } \lambda x\lambda y.\boldsymbol{\pi} xy\\
\mathbf{\Sigma}^* &\text{ is } \lambda x\lambda y.\boldsymbol{\sigma} xy\\
\mathbf{+}^* &\text{ is } \lambda x\lambda y.\overline{\boldsymbol{pl}}xy\\
\mathbf{I}^* &\text{ is } \lambda z\lambda x\lambda y.\boldsymbol{i}zxy\\
\mathbf{N}^* &\text{ is } \omega\\
\mathbf{N_k}^* &\text{ is } \bar{k}\\
\mathbf{U}^* &\text{ is } \bar{\boldsymbol{u}}\\
\mathbf{V}^* &\text{ is } \bar{\boldsymbol{v}}\\
\mathbf{s_N}^* &\text{ is } \mathbf{s_N}\\
\mathbf{r}^* &\text{ is } 0\\
\boldsymbol{\lambda}^* &\text{ is } \lambda x.x \quad \text{(i.e. } \mathbf{skk}\text{)}\\
\mathbf{ap}^* &\text{ is } \lambda x\lambda y.yx\\
\mathbf{sup}^* &\text{ is } \lambda x\lambda y.(x, y)\\
\mathbf{E}^* &\text{ is } \lambda x\lambda y.y(\mathbf{p_1}x, \mathbf{p_1}x)\\
\mathbf{i}^* &\text{ is } \lambda x.(0, x)\\
\mathbf{j}^* &\text{ is } \lambda x.(1, x)\\
\mathbf{D}^* &\text{ is } \lambda x\lambda y\lambda z.(0, \mathbf{p_1}x, y(\mathbf{p_1}x), z(\mathbf{p_1}x))\\
\mathbf{J}^* &\text{ is } \lambda x\lambda y.y.
\end{aligned}$$

$\mathbf{R_k}^*$ is $\lambda m.\lambda x_0 \ldots \lambda x_{k-1}.e_k(m, x_0, \ldots, x_{k-1})$, where an application term e_k is chosen so that $\mathbf{CZF}$ proves $e_k(m, x_0, \ldots, x_{k-1}) \simeq x_m$ if $m < k$.

$\mathbf{R}^*$ is an application term introduced by the Recursion Theorem 4.7 to satisfy $\mathbf{R}^*ab0 \simeq a$ and $\mathbf{R}^*ab(\mathbf{s_N}x) \simeq bx(\mathbf{R}^*abx)$. $\mathbf{T_V^*}$ is a term introduced by the Recursion Theorem to satisfy

$$\begin{aligned}
&\mathbf{T_V^*}(\mathbf{sup}^*(a, b), \lambda x.\lambda y.\lambda z.e(x, y, z)) \simeq\\
&\quad e(a, b, (\lambda x.x)(\lambda v.\mathbf{T_V^*}(\mathbf{ap}^*(b, v), \lambda x.\lambda y.\lambda z.e(x, y, z)))).
\end{aligned}$$

For a variable u let u^* be u. For complex terms of $\mathbf{ML_1V}$ we define:

$$\begin{aligned} ((x_1, \ldots, x_n)t)^* &\text{ is } \lambda x_1 \ldots \lambda x_n.t^*;\\ (t(s))^* &\text{ is } t^* s^*;\\ (t, s)^* &\text{ is } \mathbf{p}(t^*, s^*). \end{aligned}$$

Definition 6.5. The *type terms* of $\mathbf{ML_1V}$ are defined inductively by

1. $\mathbf{N}$ and $\mathbf{N_k}$ are type terms (for each integer k);
2. If A and B are type terms, so is $(A+B)$;
3. If $B(x)$ and A are type terms, and x is not free in A or in B, then $\mathbf{\Pi}(A, B)$ and $\mathbf{\Sigma}(A, B)$ are type terms;
4. If A is a type term and t, s are any terms of $\mathbf{ML_1V}$, then $\mathbf{I}(A, s, t)$ is a type term;
5. $\mathbf{U}$ and $\mathbf{V}$ are type terms;
6. If A is a type term and $B \stackrel{\triangle}{\equiv} A$, then B is a type term.

Definition 6.6. *Large types* (type terms) are those containing the constants $\mathbf{U}$ or $\mathbf{V}$. Others are *small types*. Another way of rendering this distinction is by saying that A is a small type iff $A : \mathbf{U}$.

Definition 6.7 (*Interpretation of* $\mathbf{ML_1V}$ *in* $\mathbf{CZF}$). By induction on the complexity of the type term A we shall assign to each judgement Φ of $\mathbf{ML_1V}$ of the form $u : A$ or $u = v : A$ (u, v variables) a formula $(\Phi)^\wedge$ of $\mathbf{CZF}$ with the same free variables. $(ux : A)^\wedge$ and $(ux = uy : A)^\wedge$ will be used as short for $\exists z[\{u\}(x) \simeq z \wedge (z : A)^\wedge]$ and $\exists z[\{u\}(x) \simeq z \wedge \{u\}(y) \simeq z \wedge (z : A)^\wedge]$, respectively. Likewise, $(u : A(vx))^\wedge$ abbreviates $\exists z[\{v\}(x) \simeq z \wedge (u : A(z))^\wedge]$, etc. Also, $(ux : A)^\wedge$ and $(ux = uy : A)^\wedge$ will be used as shorthand for $\exists z[\{u\}(x) \simeq z \wedge (z : A)^\wedge]$ and $\exists z[\{u\}(x) \simeq z \wedge \{u\}(y) \simeq z \wedge (z : A)^\wedge]$, respectively. Below we shall use $\bar{k}$ to denote the kth von Neumann integer, that is, the kth member of ω.

The clauses in the definition are as follows:

$$\begin{aligned}
(f : \mathbf{\Pi}(A, B))^\wedge \text{ is } & \mathrm{Fun}(f) \wedge \forall z \in f\,(((z)_0 : A)^\wedge \wedge ((z)_1 : B((z)_0))^\wedge) \wedge\\
& \forall x[(x : A)^\wedge \to (f(x) : B(x))^\wedge] \wedge\\
& \forall x, y[(x = y : A)^\wedge \to (f(x) = f(y) : B(x))^\wedge]\\
& \text{if } A \text{ is a small type}\\
(u : \mathbf{\Pi}(A, B))^\wedge \text{ is } & \forall x[(x : A)^\wedge \to (ux : B(x))^\wedge] \wedge\\
& \forall x, y[(x = y : A)^\wedge \to (ux = uy : B(x))^\wedge]\\
& \text{if } A \text{ is a large type}\\
(u = v : \mathbf{\Pi}(A, B))^\wedge \text{ is } & (u : \mathbf{\Pi}(A, B))^\wedge \wedge (v : \mathbf{\Pi}(A, B))^\wedge \wedge\\
& \forall x[(x : A)^\wedge \to (ux = vx : B(x))^\wedge]\\
& \text{if } A \text{ is a small type}\\
(u = v : \mathbf{\Pi}(A, B))^\wedge \text{ is } & (u : \mathbf{\Pi}(A, B))^\wedge \wedge (v : \mathbf{\Pi}(A, B))^\wedge \wedge\\
& \forall x[(x : A)^\wedge \to (ux = vx : B(x))^\wedge]\\
& \text{if } A \text{ is a large type}\\
(u : \mathbf{\Sigma}(A, B))^\wedge \text{ is } & u = \langle (u)_0, (u)_1 \rangle \wedge ((u)_0 : A)^\wedge \wedge ((u)_1 : B((u)_0))^\wedge\\
(u = v : \mathbf{\Sigma}(A, B))^\wedge \text{ is } & (u : \mathbf{\Sigma}(A, B))^\wedge \wedge (v : \mathbf{\Sigma}(A, B))^\wedge \wedge\\
& ((u)_0 = (v)_0 : A)^\wedge \wedge ((u)_1 = (v)_1 : B((u)_0))^\wedge\\
(u : (A+B))^\wedge \text{ is } & u = \langle (u)_0, (u)_1 \rangle \wedge [[(u)_0 = 0 \wedge ((u)_1 : A)^\wedge] \vee\\
& [(u)_0 = 1 \wedge ((u)_1 : B)^\wedge]]\\
(u = v : (A+B))^\wedge \text{ is } & (u : (A+B))^\wedge \wedge (v : (A+B))^\wedge \wedge\\
& \big[[(u)_0 = 0 \wedge (v)_0 = 0 \wedge ((u)_1 = (v)_1 : A)^\wedge] \vee\\
& [(u)_0 = 1 \wedge (v)_0 = 1 \wedge ((u)_1 = (v)_1 : B)^\wedge]\big]\\
(u : \mathbf{I}(A, b, c))^\wedge \text{ is } & u = 0 \wedge (b = c : A)^\wedge
\end{aligned}$$

$$
\begin{aligned}
(u = v : \mathbf{I}(A, b, c))^\wedge &\quad \text{is} \quad u = 0 \wedge v = 0 \wedge (b = c : A)^\wedge \\
(u : \mathbf{N})^\wedge &\quad \text{is} \quad u \in \omega \\
(u = v : \mathbf{N})^\wedge &\quad \text{is} \quad u = v \wedge u \in \omega \\
(u : \mathbf{N_k})^\wedge &\quad \text{is} \quad u \in \bar{k} \\
(u = v : \mathbf{N_k})^\wedge &\quad \text{is} \quad u = v \wedge u \in \bar{k} \\
(u : \mathbf{U})^\wedge &\quad \text{is} \quad u \in \mathbf{Y}^* \\
(u = v : \mathbf{U})^\wedge &\quad \text{is} \quad u = v \wedge u \in \mathbf{Y}^* \wedge v \in \mathbf{Y}^* \\
(u : \mathbf{V})^\wedge &\quad \text{is} \quad u \in \mathbf{V}(\mathbf{Y}^*) \\
(u = v \in \mathbf{V})^\wedge &\quad \text{is} \quad u = v \wedge u \in \mathbf{V}(\mathbf{Y}^*) \wedge v \in \mathbf{V}(\mathbf{Y}^*)
\end{aligned}
$$

If s and t are arbitrary terms of $\mathbf{ML_1V}$ and A is a type term of $\mathbf{ML_1V}$, we set:

$$
\begin{aligned}
(t : A)^\wedge &\quad \text{is} \quad \exists u[t^* \simeq u \wedge (u : A)^\wedge], \\
(s = t : A)^\wedge &\quad \text{is} \quad \exists u, v[s^* \simeq u \wedge t^* \simeq v \wedge (u = v : A)^\wedge].
\end{aligned}
$$

For type terms A and B we define $(A = B)^\wedge$ by

$$\forall u[(u : A)^\wedge \leftrightarrow (u : B)^\wedge] \wedge \forall u, v[(u = v : A)^\wedge \leftrightarrow (u = v : B)^\wedge].$$

In the natural deduction style presentation of type theory one deduces hypothetical judgements, i.e. judgements which are made under assumptions (see [14] pp. 16–20). We shall use the notation

$$\mathbf{ML_1V} \vdash \Phi(u_1, \ldots, u_n)\ [u_1 : A_1, \ldots, u_n : A(u_1, \ldots, u_{n-1})]$$

to convey that the judgement $\Phi(u_1, \ldots, u_n)$ is deducible in $\mathbf{ML_1V}$ under the open assumptions $u_1 : A_1, \ldots, u_n : A(u_1, \ldots, u_{n-1})$.

Theorem 6.8 (*Soundness of the Interpretation of* $\mathbf{ML_1V}$ *in* **CZF**). *If*

$$\mathbf{ML_1V} \vdash \Phi(u_1, \ldots, u_n)\ [u_1 : A_1, \ldots, u_n : A(u_1, \ldots, u_{n-1})],$$

where $\Phi(u_1, \ldots, u_n)$ is a judgement not of the form "A type", then

$$\mathbf{CZF} \vdash (u_1 : A_1)^\wedge \wedge \ldots \wedge (u_n : A(u_1, \ldots, u_{n-1}))^\wedge \rightarrow (\Phi(u_1, \ldots, u_n))^\wedge.$$

Proof. First note that if an expression of the form "A *type*", $s : A$, $s = t : A$, or $A = B$ appears in a derivation of $\mathbf{ML_1V}$, then A is a type term in the sense of Definition 6.5, as is readily seen by induction on derivations in $\mathbf{ML_1V}$. This ensures that any judgment of $\mathbf{ML_1V}$ gets translated under $^\wedge$. Secondly, it should be clear that the above interpretation replaces the abstract application of $\mathbf{ML_1V}$ by set-recursive application in a faithful way, i.e. the equations which the rules of $\mathbf{ML_1V}$ prescribe for the constants of $\mathbf{ML_1V}$ are satisfied by their translations. The constructions 2.8 and 3.1 ensure that particular rules for **U** and **V**-introduction are sound with respect to the interpretation $^\wedge$. The soundness of **V**-elimination and **V**-equality is verified in the same way as in [19, Th. 4.11]. □

The foregoing interpretation can be extended to $\mathbf{ML_1WV}$. $\mathbf{ML_1WV}$ has the additional constants **W** and $\mathbf{T_W}$, where $\mathbf{T_W}$ is the eliminatory constant associated with the **W**-type. $\mathbf{ML_1WV}$ has additional type terms of the form $\mathbf{W}(A, B)$ providing A is a small type and $B(x)$ is a small type for every $x : A$. Here a small type is one that does not involve **U** or **V** (but may contain **W**).

The translation of 6.4 has to be altered for the types **U** and **V** as follows

$$
\begin{aligned}
(u : \mathbf{U})^\wedge &\quad \text{is} \quad u \in \mathbf{Y_w^*} \\
(u = v : \mathbf{U})^\wedge &\quad \text{is} \quad u = v \wedge u \in \mathbf{Y_w^*} \wedge v \in \mathbf{Y_w^*} \\
(u : \mathbf{V})^\wedge &\quad \text{is} \quad u \in \mathbf{V}(\mathbf{Y_w^*}) \\
(u = v \in \mathbf{V})^\wedge &\quad \text{is} \quad u = v \wedge u \in \mathbf{V}(\mathbf{Y_w^*}) \wedge v \in \mathbf{V}(\mathbf{Y_w^*})
\end{aligned}
$$

and is to be continued to terms of $\mathbf{ML_{1W}V}$ by letting $\mathbf{W}^*$ be $\lambda x \lambda y.\bar{\mathbf{w}}xy$ and $\mathbf{T}^*_{\mathbf{W}}$ be defined similar to $\mathbf{T}^*_{\mathbf{V}}$ in 6.4. Next, building on 6.7, we need to translate judgements of the form $u : \mathbf{W}(A, B)$ and $u = v : \mathbf{W}(A, B)$.

$$(u : \mathbf{W}(A, B))^\wedge \quad \text{is} \quad \mathbf{W}_{z \in A^*} B^*(z),$$

where $A^* = \{x \mid (x : A)^\wedge\}$ and the function B^* with domain A^* is defined by $B^*(x) = \{z \mid (z : B(x))^\wedge\}$.

$$\begin{aligned}(u = v : \mathbf{W}(A, B))^\wedge \quad \text{is} \quad & (u : \mathbf{W}(A, B))^\wedge \wedge (v : \mathbf{W}(A, B))^\wedge \wedge \\ & ((u)_0 = (v)_0 : A)^\wedge \wedge \\ & \forall x[(x : A)^\wedge \to ((u)_1(x) = (v)_1(x) : B(x))^\wedge].\end{aligned}$$

The interpretation of $\mathbf{ML_1V}$ in **CZF** given in 6.8 can then be extended as follows.

Theorem 6.9 (*Soundness of the Interpretation of* $\mathbf{ML_{1W}V}$ *in* $\mathbf{CZF} + \mathbf{REA}$). *If*

$$\mathbf{ML_{1W}V} \vdash \Phi(u_1, \ldots, u_n)\ [u_1 : A_1, \ldots, u_n : A(u_1, \ldots, u_{n-1})],$$

where $\Phi(u_1, \ldots, u_n)$ *is a judgement not of the form "A* type*", then*

$$\mathbf{CZF} + \mathbf{REA} \vdash (u_1 : A_1)^\wedge \wedge \ldots \wedge (u_n : A(u_1, \ldots, u_{n-1}))^\wedge \to (\Phi(u_1, \ldots, u_n))^\wedge.$$

7. Combining the interpretations

By now, several interpretations among set theories and between set theories and type theories have accrued, and we may combine them to characterize the formulae validated in type theory.

[1–3] provide interpretations of $\mathbf{CZF} + \mathbf{\Pi\Sigma}{-}\mathbf{AC}$ and $\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W}{-}\mathbf{AC}$ in Martin-Löf's type theories $\mathbf{ML_1V}$ and $\mathbf{ML_{1W}V}$, respectively. Conversely, using Theorems 6.8 and 6.9, we have interpretations of $\mathbf{ML_1V}$ and $\mathbf{ML_{1W}V}$ in $\mathbf{CZF} + \mathbf{\Pi\Sigma} - \mathbf{AC}$ and $\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W} - \mathbf{AC}$, respectively. Specifically, if θ is a set-theoretic sentence, and $\mathbf{CZF} \vdash \theta$, then $\mathbf{ML_1V} \vdash t : \|\theta\|$ for some term t. On the other hand, assuming $\mathbf{ML_1V} \vdash t : \|\theta\|$ we arrive at $\mathbf{CZF} \vdash (t : \|\theta\|)^\wedge$, owing to Theorem 6.8. It will be shown that this entails the inhabitedness of $[\![\theta]\!]$ provably in **CZF**. Whence, if θ is a mathematical formula we can utilize Corollary 5.20 to conclude that $\mathbf{CZF} + \mathbf{\Pi\Sigma}{-}\mathbf{AC} \vdash \theta$.

Definition 7.1 (*See [7, XII.1.4]*). A **V**-*assignment* is a function $\mathcal{M} : \mathrm{Var} \to \mathbf{V}$. For every formula φ of **CZF** and **V**-assignment $\mathcal{M}$, we define a type term $\|\varphi\|_{\mathcal{M}}$ of $\mathbf{ML_1V}$. First, for $\alpha, \beta : \mathbf{V}$, recall that $\|\alpha = \beta\|$ is defined by recursion on **V** and equates to

$$\Pi(\bar{\alpha}, \lambda x.\Sigma(\bar{\beta}, \lambda y.\|\tilde{\alpha}(x) = \tilde{\beta}(y)\|)) \times \Pi(\bar{\beta}, \lambda x.\Sigma(\bar{\alpha}, \lambda y.\|\tilde{\beta}(x) = \tilde{\alpha}(y)\|)).$$

The rest of the definition is as follows:

$$\begin{array}{lll}
\|u \in v\|_{\mathcal{M}} & \text{is} & \Sigma(\bar{\beta}, \lambda y.\|\alpha = \tilde{\beta}(y)\|) \text{ where } \alpha = \mathcal{M}(u), \beta = \mathcal{M}(v), \\
\|\varphi_0 \wedge \varphi_1\|_{\mathcal{M}} & \text{is} & \|\varphi_0\|_{\mathcal{M}} \times \|\varphi_1\|_{\mathcal{M}} \\
\|\varphi_0 \vee \varphi_1\|_{\mathcal{M}} & \text{is} & \|\varphi_0\|_{\mathcal{M}} + \|\varphi_1\|_{\mathcal{M}} \\
\|\varphi_0 \supset \varphi_1\|_{\mathcal{M}} & \text{is} & \|\varphi_0\|_{\mathcal{M}} \to \|\varphi_1\|_{\mathcal{M}} \\
\|\bot\|_{\mathcal{M}} & \text{is} & \mathbf{N_0} \\
\|\forall u \in \alpha\varphi\|_{\mathcal{M}} & \text{is} & \Pi\big(\bar{\alpha}, \lambda x.\|\varphi\|_{\mathcal{M}(u|\tilde{\alpha}(x))}\big) \\
\|\exists u \in \alpha\varphi\|_{\mathcal{M}} & \text{is} & \Sigma\big(\bar{\alpha}, \lambda x.\|\varphi\|_{\mathcal{M}(u|\tilde{\alpha}(x))}\big) \\
\|\forall u\varphi\|_{\mathcal{M}} & \text{is} & \Pi\big(\mathbf{V}, \lambda\alpha.\|\varphi\|_{\mathcal{M}(u|\alpha)}\big) \\
\|\exists u\varphi\|_{\mathcal{M}} & \text{is} & \Sigma\big(\mathbf{V}, \lambda\alpha.\|\varphi\|_{\mathcal{M}(u|\alpha)}\big).
\end{array}$$

If φ is a set-theoretic formula whose free variables are among $u_1, \ldots, u_n, \alpha_1, \ldots, \alpha_n : \mathbf{V}$, and the **V**-assignment $\mathcal{M}$ satisfies $\mathcal{M}(v_i) = \alpha_i$ for $i = 1, \ldots, n$ we also write "$\|\varphi(\alpha_1, \ldots, \alpha_n)\|$" for "$\|\varphi\|_{\mathcal{M}}$".

It is easy to prove by induction on the complexity of φ that $\|\varphi\|_{\mathcal{M}}$ is a type for all formulae φ, and a small type for bounded φ.

We **note** that, according to the constructions 2.8 and 3.1, the type $\mathbf{U}$ of $\mathbf{ML_1V}$ can be identified with the class $\mathbf{Y}^*$ of sets, and the type $\mathbf{V}$ can be identified with the class $\mathbf{V}(\mathbf{Y}^*)$. This in particular means that the small types, i.e., those of Martin-Löf's types belonging to $\mathbf{U}$, have their set-theoretic counterpart in $\mathbf{Y}^*$. This will enable us to identify a $\mathbf{V}$-assignment with a $\mathbf{V}(\mathbf{Y}^*)$-assignment.

Likewise, owing to the constructions 2.9 and 3.3, the type $\mathbf{U}$ of $\mathbf{ML_{1W}V}$ can also be identified with the class $\mathbf{Y}^*_\mathbf{w}$ of sets, and the type $\mathbf{V}$ of $\mathbf{ML_{1W}V}$ can be identified with the class $\mathbf{V}(\mathbf{Y}^*_\mathbf{w})$. This in particular means that the small types of $\mathbf{ML_{1W}V}$ have their set-theoretic counterparts in $\mathbf{Y}^*_\mathbf{w}$. In this sense we will identify a $\mathbf{V}$-assignment with a $\mathbf{V}(\mathbf{Y}^*_\mathbf{w})$-assignment.

Lemma 7.2 (**CZF**). *For every set-theoretic formula φ whose free variables are among $u_1, \ldots, u_n$ and $\alpha_1, \ldots, \alpha_n \in \mathbf{V}(\mathbf{Y}^*)$, $(x : \|\varphi(\alpha_1, \ldots, \alpha_n)\|)^\wedge$ implies $x \in [\![\varphi(\alpha_1, \ldots, \alpha_n)]\!]$.*

Proof. This follows by induction on the complexity of φ, comparing 4.10 and 7.1 and the translation 6.7. □

Lemma 7.3 (**CZF + REA**). *For every set-theoretic formula φ whose free variables are among $u_1, \ldots, u_n$ and $\alpha_1, \ldots, \alpha_n \in \mathbf{V}(\mathbf{Y}^*_\mathbf{w})$, $(x : \|\varphi(\alpha_1, \ldots, \alpha_n)\|)^\wedge$ implies $x \in [\![\varphi(\alpha_1, \ldots, \alpha_n)]\!]$.*

Proof. Similar to 7.2. □

Theorem 7.4. *If φ is a formula in* CC *with at most $u_1, \ldots, u_n$ free and*

$$\mathbf{ML_1V} \vdash t : \|\varphi(\vec{\alpha})\| \; [\alpha_1 : \mathbf{V}, \ldots, \alpha_n : \mathbf{V}]$$

for some term t (where $\vec{\alpha} = \alpha_1, \ldots, \alpha_n$) then

$$\mathbf{CZF} + \mathbf{\Pi\Sigma{-}AC} \vdash \vec{\alpha} \in \mathbf{V}(\mathbf{Y}^*) \;\to\; \varphi\big(\ell(\alpha_1), \ldots, \ell(\alpha_n)\big)^{\mathbf{H}(\mathbf{Y}^*)}.$$

Proof. By Theorem 6.8 we have

$$\mathbf{CZF} \vdash \vec{\alpha} \in \mathbf{V}(\mathbf{Y}^*) \;\to\; \exists u((t^* \simeq u)^\wedge \wedge (u : \|\varphi(\vec{\alpha})\|)^\wedge).$$

By Lemma 7.2 we get

$$\mathbf{CZF} \vdash \alpha_1, \ldots, \alpha_n \in \mathbf{V}(\mathbf{Y}^*) \;\to\; \exists u \left(u \in [\![\varphi(\alpha_1, \ldots, \alpha_n)]\!]\right)$$

which by Theorem 5.2 implies the desired assertion. □

Theorem 7.5. *If φ is a generalized mathematical formula with at most $u_1, \ldots, u_n$ free and*

$$\mathbf{ML_{1W}V} \vdash t : \|\varphi(\alpha_1, \ldots, \alpha_n)\| \;\; [\alpha_1 : \mathbf{V}, \ldots, \alpha_n : \mathbf{V}]$$

for some term t, then, letting $\ell_\mathbf{w}(\vec{\alpha}) = \ell_\mathbf{w}(\alpha_1), \ldots, \ell_\mathbf{w}(\alpha_n)$,

$$\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W{-}AC} \vdash \vec{\alpha} \in \mathbf{V}(\mathbf{Y}^*_\mathbf{w}) \;\to\; \varphi\big(\ell_\mathbf{w}(\vec{\alpha})\big)^{\mathbf{H}(\mathbf{Y}^*_\mathbf{w})}. \tag{21}$$

Proof. By Theorem 6.9 we have

$$\mathbf{CZF} + \mathbf{REA} \vdash \vec{\alpha} \in \mathbf{V}(\mathbf{Y}^*_\mathbf{w}) \;\to\; \exists u((t^* \simeq u)^\wedge \wedge (u : \|\varphi(\vec{\alpha})\|)^\wedge).$$

By Lemma 7.3 we get

$$\mathbf{CZF} + \mathbf{REA} \vdash \vec{\alpha} \in \mathbf{V}(\mathbf{Y}^*_\mathbf{w}) \;\to\; \exists u \left(u \in [\![\varphi(\vec{\alpha})]\!]\right)$$

which by Theorem 5.16 implies (21). □

Theorem 7.6. *Let ψ be a mathematical sentence and let θ be a generalized mathematical sentence. Then the following hold:*

(i) $\mathbf{CZF} + \mathbf{\Pi\Sigma{-}AC} \vdash \psi$ *if and only if* $\mathbf{ML_1V} \vdash t_\psi : \|\psi\|$ *for some term t_ψ of* $\mathbf{ML_1V}$.
(ii) $\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W{-}AC} \vdash \theta$ *if and only if* $\mathbf{ML_{1W}V} \vdash t_\theta : \|\theta\|$ *for some term t_θ of* $\mathbf{ML_{1W}V}$.

Proof. The directions “⇒” follow by inspection of the proofs in [1–3]. Now suppose that $\mathbf{ML_1V} \vdash t_\psi : \|\psi\|$ for some term t_ψ of $\mathbf{ML_1V}$. By 5.8, ψ is a CC-formula so that by 7.4 we arrive at $\mathbf{CZF} + \mathbf{\Pi\Sigma{-}AC} \vdash \psi^{\mathbf{H(Y^*)}}$, whence $\mathbf{CZF} + \mathbf{\Pi\Sigma{-}AC} \vdash \psi$ owing to 5.20(i).

Next assume $\mathbf{ML_1WV} \vdash t_\theta : \|\theta\|$. Then 7.5 yields $\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W} - \mathbf{AC} \vdash \theta^{\mathbf{H(Y^*_w)}}$, so that $\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W{-}AC} \vdash \theta$ follows from 5.20(ii). □

8. The existence property

It is often considered a hallmark of intuitionistic systems that they possess the disjunction and existential definability properties.

Definition 8.1. Let T be a theory whose language, $L(T)$, encompasses the language of set theory. Moreover, for simplicity, we shall assume that $L(T)$ has a constant ω denoting the set of von Neumann natural numbers and for each n a constant $\bar{n}$ denoting the nth natural number.

1. T has the *disjunction property*, **DP**, if whenever $\psi \vee \theta$ is closed and $T \vdash \psi \vee \theta$ then $T \vdash \psi$ or $T \vdash \theta$.
2. T has the *numerical existence property*, **NEP**, if whenever $(\exists x \in \omega)\phi(x)$ is closed and $T \vdash (\exists x \in \omega)\phi(x)$ then $T \vdash \phi(\bar{n})$ for some n.
3. T has the *existence property*, **EP**, if whenever $\exists x \phi(x)$ is closed and $T \vdash \exists x \phi(x)$ then $T \vdash \exists! x\,[\vartheta(x) \wedge \phi(x)]$ for some formula ϑ with no free variables other than x.

Slightly abusing terminology, we shall also say that T enjoys any of these properties if this holds only for a definitional extension of T rather than T.

ZF and **ZFC** do not have the existence property. But even classical set theories can have the **EP**. Kunen observed that an extension of **ZF** has the **EP** if and only if it proves that all sets are ordinal definable, i.e., $V = OD$. Going back to intuitionistic set theories, let $\mathbf{IZF}_R$ result from **IZF** by replacing Collection with Replacement, and let **CST** be Myhill’s constructive set theory of [17]. Also let $\mathbf{CST}^-$ be **CST** without the axioms of countable and dependent choice.

Theorem 8.2. (i) $\mathbf{IZF}_R$ *and* $\mathbf{CST}^-$ *have the* **DP**, **NEP**, *and the* **EP**. **CST** *has the* **DP** *and* **NEP**.
(ii) **IZF** *has the* **DP** *and the* **NEP**.
(iii) **IZF** *does not have the* **EP**.
(iv) **CZF** *and* $\mathbf{CZF} + \mathbf{REA}$ *have the* **DP** *and the* **NEP**.

Proof. (i) is proved in [17]. For (ii) see [6] and for (iii) see [12]. (iv) is [22], Theorem 1.2. □

The question of whether **CZF** satisfies the existence property is currently unanswered. Friedman’s proof of the failure of **EP** for **IZF** seems to single out Collection as the culprit. However, that proof does not seem to carry over to **CZF** since the refutation of **EP** uses existential statements of the form

$$\exists b \left[\forall u \in a\, \exists y\, \varphi(u, y) \;\rightarrow\; \forall u \in a\, \exists y \in b\, \varphi(u, y)\right],$$

which are deducible in **IZF** by employing Collection and full Separation, but need not be deducible in **CZF**. The first author conjectures that **EP** fails for **CZF** on account of Subset Collection (and maybe Collection). There are, however, positive answers available for $\mathbf{CZF} + \mathbf{\Pi\Sigma{-}AC}$ and $\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W{-}AC}$ in that these theories can be shown to have the **EP** for mathematical and generalized mathematical statements, respectively.

Theorem 8.3. *Let θ_1, θ_2 be $\Sigma(math)$ sentences and let $\psi(x)$ be a $\Sigma(math)$ formula with at most x free. Then we have the following:*

(i) *If* $\mathbf{CZF} + \mathbf{RDC} + \mathbf{\Pi\Sigma{-}AC} \vdash \theta_1 \vee \theta_2$ *then* $\mathbf{CZF} + \mathbf{\Pi\Sigma{-}AC} \vdash \theta_1$ *or* $\mathbf{CZF} + \mathbf{\Pi\Sigma{-}AC} \vdash \theta_2$.
(ii) *If* $\mathbf{CZF}{+}\mathbf{RDC}{+}\mathbf{\Pi\Sigma{-}AC} \vdash \exists u \in \omega\, \psi(u)$ *then there exists a natural number n such that* $\mathbf{CZF}{+}\mathbf{\Pi\Sigma{-}AC} \vdash \psi(\bar{n})$.
(iii) *If* $\mathbf{CZF} + \mathbf{RDC} + \mathbf{\Pi\Sigma} - \mathbf{AC} \vdash \exists x \psi(x)$ *then there is a formula* $\vartheta(x)$ *(with at most x free) such that* $\mathbf{CZF} + \mathbf{\Pi\Sigma{-}AC} \vdash \exists! x[\vartheta(x) \wedge \psi(x)]$.

Proof. (i): Suppose $\mathbf{CZF}+\mathbf{RDC}+\mathbf{\Pi\Sigma{-}AC}\vdash\theta_1\vee\theta_2$. By [21], Theorem 4.14 one can (primitive recursively) find a closed application term t such that $\mathbf{CZF}\vdash\exists x\,\big[t\simeq x\ \wedge\ x\in[\![\theta_1\vee\theta_2]\!]\big]$ so that

$$\mathbf{CZF}\vdash\exists i\in\omega\,\big([i=0\ \wedge\ \exists u\,u\in[\![\theta_1]\!]]\ \vee\ [i=1\ \wedge\ \exists u\,u\in[\![\theta_2]\!]]\big).$$

As $\mathbf{CZF}$ has the numerical existence property by Theorem 8.2(iv), this implies $\mathbf{CZF}\vdash\exists u\,u\in[\![\theta_1]\!]$ or $\mathbf{CZF}\vdash\exists u\,u\in[\![\theta_2]\!]$, whence by Theorem 5.21 (i), $\mathbf{CZF}+\mathbf{\Pi\Sigma{-}AC}\vdash\theta_1$ or $\mathbf{CZF}+\mathbf{\Pi\Sigma{-}AC}\vdash\theta_2$.
(ii): Suppose $\mathbf{CZF}+\mathbf{RDC}+\mathbf{\Pi\Sigma{-}AC}\vdash\exists u\in\omega\,\psi(u)$. By [21], Theorem 4.14 one can (primitive recursively) find a closed application term t' such that $\mathbf{CZF}\vdash\exists x\,\big(t'\simeq x\ \wedge\ x\in[\![\exists u\in\omega\,\psi(u)]\!]\big)$. At this point we have to go back to the details of the proof of [21], Lemma 4.17. The role of ω in $\mathbf{V}(\mathbf{Y}^*)$ is played by $\omega^*=\sup(\omega,h_\omega)$, where $h_\omega:\omega\to\mathbf{V}(\mathbf{Y}^*)$. We then obtain $\mathbf{CZF}\vdash\exists y\,\big(t\simeq y\ \wedge\ y\in[\![\exists u\in\omega^*\psi(u)]\!]\big)$ for a closed application term t, and thence $\mathbf{CZF}\vdash\exists i\in\omega\,\exists z\,z\in[\![\psi(h_\omega(i))]\!]$. Since $\mathbf{CZF}$ enjoys the $\mathbf{NEP}$, there exists a natural number n such that $\mathbf{CZF}\vdash\exists z\,z\in[\![\psi(h_\omega(\bar n))]\!]$. It also follows from the definition of h_ω (cf. [21], 4.14) that $\ell(h_\omega(\bar n))=\bar n$. Thus, by Theorem 5.21(i), $\mathbf{CZF}+\mathbf{\Pi\Sigma{-}AC}\vdash\psi(\bar n)$.
(iii): Now suppose $\mathbf{CZF}+\mathbf{RDC}+\mathbf{\Pi\Sigma{-}AC}\vdash\exists x\,\psi(x)$. Then, owing to [21], Theorem 4.14, one can (primitive recursively) find a closed application term t such that $\mathbf{CZF}\vdash\exists z\,\big[t\simeq z\ \wedge\ z\in[\![\exists x\,\psi(x)]\!]\big]$ so that

$$\mathbf{CZF}\vdash\exists\alpha\in\mathbf{V}(\mathbf{Y}^*)\,\big[\mathbf{p_0}t\simeq\alpha\ \wedge\ \mathbf{p_1}t\in[\![\psi(\alpha)]\!]\big].$$

By 5.21(i) the latter yields

$$\mathbf{CZF}+\mathbf{\Pi\Sigma{-}AC}\vdash\exists\alpha\in\mathbf{V}(\mathbf{Y}^*)\,\big[\mathbf{p_0}t\simeq\alpha\ \wedge\ \psi(\ell(\alpha))\big].$$

Now define $\vartheta(x)$ by $\exists\alpha\in\mathbf{V}(\mathbf{Y}^*)\,\big[\mathbf{p_0}t\simeq\alpha\ \wedge\ x=\ell(\alpha)\big]$. Then $\mathbf{CZF}+\mathbf{\Pi\Sigma{-}AC}\vdash\exists!x[\vartheta(x)\wedge\psi(x)]$. □

Theorem 8.4. *Let θ_1,θ_2 be $\Sigma(gmath)$ sentences and let $\psi(x)$ be a $\Sigma(gmath)$ formula with at most x free. Then we have the following:*

(i) *If* $\mathbf{CZF}+\mathbf{REA}+\mathbf{RDC}+\mathbf{\Pi\Sigma W{-}AC}\vdash\theta_1\vee\theta_2$ *then* $\mathbf{CZF}+\mathbf{REA}+\mathbf{\Pi\Sigma W{-}AC}\vdash\theta_1$ *or* $\mathbf{CZF}+\mathbf{REA}+\mathbf{\Pi\Sigma W{-}AC}\vdash\theta_2$.
(ii) *If* $\mathbf{CZF}+\mathbf{REA}+\mathbf{RDC}+\mathbf{\Pi\Sigma W{-}AC}\vdash\exists u\in\omega\,\psi(u)$ *then there exists a natural number n such that* $\mathbf{CZF}+\mathbf{REA}+\mathbf{\Pi\Sigma W{-}AC}\vdash\psi(\bar n)$.
(iii) *If* $\mathbf{CZF}+\mathbf{REA}+\mathbf{RDC}+\mathbf{\Pi\Sigma W{-}AC}\vdash\exists x\,\psi(x)$ *then there is a formula $\vartheta(x)$ (with at most x free) such that* $\mathbf{CZF}+\mathbf{REA}+\mathbf{\Pi\Sigma W{-}AC}\vdash\exists!x[\vartheta(x)\wedge\psi(x)]$.

Proof. The proof results from that of 8.3 by replacing the reference to [21] 4.14 by reference to [21] 4.33, and using 5.21(ii) in place of 5.21(i). □

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